Finite Difference Time Domain Method (1)

All electromagnetic calculations can be done in either the time domain or the frequency domain. The time domain quantities are related to the frequency domain quantities by the Fourier transform.

\[
\tilde{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t)e^{-j\omega t} \, dt \quad \text{and} \quad \tilde{E}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(\omega)e^{+j\omega t} \, d\omega
\]

For example, in the frequency domain the far field radiation due to currents is

\[
\tilde{E}(\omega) = -\frac{j\omega\mu_0}{4\pi} \int_{V} \left\{ \tilde{J}(\omega) - \left[ \tilde{J}(\omega) \cdot \hat{r} \right] \hat{r} \right\} e^{-jkr} \frac{dr}{r} \, dv'
\]

Define the retarded current

\[
\tilde{J}(t - r/c) \equiv \int_{-\infty}^{t} \tilde{J}(\omega)e^{j\omega(t - r/c)} \, d\omega
\]

The time-domain expression for the electric field is

\[
\tilde{E}(t) = -\frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \left\{ \tilde{J}(t - r/c) - \left[ \tilde{J}(t - r/c) \cdot \hat{r} \right] \hat{r} \right\} \frac{dr}{r} \, dv'
\]
Transmission Through a Panel: TD vs. FD

**TIME DOMAIN**

**FREQUENCY DOMAIN**

Fourier transform →

Monocycle Waveform, Period = 1 ns

Normalized Amplitude

Time, ns

Spectrum of Monocycle

Normalized Spectrum

Frequency, GHz

Transmitted Waveform

Normalized Amplitude

Time, ns

Spectrum of Transmitted Signal

Normalized Spectrum

Frequency, GHz

Plane wave scattering by a panel

$H(\omega) \sim \Gamma(\omega)$

$\vec{E}_i(\omega)$ → $\vec{E}_s(\omega)$

$\hat{k}_i$ → $\hat{k}_s$

Linear system model:

$x(t)$

$\rightarrow$

$h(t)$

$\rightarrow$

$y(t)$

$X(\omega)$

$\rightarrow$

$H(\omega)$

$\rightarrow$

$Y(\omega)$

← Inverse Fourier transform
Finite Difference Time Domain Method (2)

The most commonly used solution methods in the time domain (TD) are physical optics and the finite difference (FD) method. For FD the spatial and time derivatives in Maxwell’s equations are approximated by finite differences. For example,

\[ \frac{\partial E}{\partial t} \rightarrow \frac{E(t + \Delta t) - E(t)}{\Delta t} \quad \text{and} \quad \frac{\partial E}{\partial x} \rightarrow \frac{E(x + \Delta x) - E(x)}{\Delta x} \]

where \( \Delta t \) is the time step and \( \Delta x \) the grid step in the \( x \) direction. The solution process is based on discretizing the target space and stepping in time as follows:

1. Derive update equations for the fields at a node in terms of the fields at the same and adjacent nodes at the current time step and previous time steps.

2. Compute the scattered fields \( \vec{E}_s(t), \vec{H}_s(t) \) over the grid boundary for a sufficient observation time \( T_{\text{max}} \)

3. Fourier transform the scattered fields to get the frequency domain fields \( \vec{E}_s(\omega), \vec{H}_s(\omega) \)

4. Find the equivalent surface currents \( \vec{J}_s(\omega), \vec{J}_{ms}(\omega) \) on the grid boundary using the computed fields

5. Apply the radiation integrals to get the far scattered fields
Finite Difference Time Domain Method (3)

Example of spatial discretization using parallelepipeds ("bricks") is shown.

The computational grid extends beyond the target.

FDTD has the advantage of adaptability. Each cell can have a permittivity and permeability independent of others.

The incident plane wave must be introduced into the grid and propagated to the target in discrete time steps.

The scattered field must exit the grid without introducing artificial reflections ("grid noise").
Finite Difference Time Domain Method (4)

Example of applying finite differences to Maxwell’s first equation: \( \frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 \).

Write the vector equation as three scalar equations in Cartesian coordinates:

\[
- \frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \quad - \frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \quad \frac{\partial B_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}
\]

Use the following shorthand notation:

Grid dimensions: \((\Delta x, \Delta y, \Delta z)\)

Time step: \(\Delta t\)

Nodes: \((i, j, k) \equiv (i\Delta x, j\Delta y, k\Delta z)\)

Fields samples:

\(E_x(i\Delta x, j\Delta y, k\Delta z, n\Delta t) \equiv E_x^n(i, j, k)\)

The \(E\) components are in the middle of the edges. The \(B\) components are in the centers of the faces, denoted by integers plus one half (for example \(i + \frac{1}{2}\)).
Finite Difference Time Domain Method (5)

The grid size must be such that over one increment the electromagnetic field does not change significantly (usually a fraction of a wavelength).

For computational stability the Courant condition should be satisfied:

$$\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} > c \Delta t = \Delta t \sqrt{\frac{1}{\mu \varepsilon}}$$

In other words, the time step cannot be chosen independently of the grid size. If sampling guidelines are violated, aliasing can occur and “numerical artifacts” appear.

Anisotropic and inhomogeneous materials can be modeled by letting $\mu, \varepsilon$ and $\sigma$ vary from cell to cell. For example, $\bar{\varepsilon}(i,j,k)$ where

$$\bar{\varepsilon}(i,j,k) = \begin{bmatrix} \varepsilon_x(i,j,k) & 0 & 0 \\ 0 & \varepsilon_y(i,j,k) & 0 \\ 0 & 0 & \varepsilon_z(i,j,k) \end{bmatrix}$$
Finite Difference Time Domain Method (6)

Magnetic field update equations:

\[ H_x^{n+1/2}(i, j, k) \approx H_x^{n-1/2}(i, j, k) + \frac{\Delta t}{\mu_x(i, j, k)} \begin{bmatrix} E_y^n(i, j, k + 1) - E_y^n(i, j, k) \\ \Delta z \end{bmatrix} - \frac{\Delta t}{\mu_x(i, j, k)} \begin{bmatrix} E_z^n(i, j + 1, k) - E_z^n(i, j, k) \\ \Delta y \end{bmatrix} \]

\[ H_y^{n+1/2}(i, j, k) \approx H_y^{n-1/2}(i, j, k) + \frac{\Delta t}{\mu_y(i, j, k)} \begin{bmatrix} E_z^n(i, j, k + 1) - E_z^n(i, j, k) \\ \Delta z \end{bmatrix} - \frac{\Delta t}{\mu_y(i, j, k)} \begin{bmatrix} E_x^n(i, j, k + 1) - E_x^n(i, j, k) \\ \Delta y \end{bmatrix} \]

\[ H_z^{n+1/2}(i, j, k) \approx H_z^{n-1/2}(i, j, k) + \frac{\Delta t}{\mu_z(i, j, k)} \begin{bmatrix} E_x^n(i, j, k + 1) - E_x^n(i, j, k) \\ \Delta y \end{bmatrix} - \frac{\Delta t}{\mu_z(i, j, k)} \begin{bmatrix} E_y^n(i, j + 1, k) - E_y^n(i, j, k) \\ \Delta x \end{bmatrix} \]

Electric field update equations:

\[ E_x^{n+1}(i, j, k) \approx K_x(i, j, k)E_x^n(i, j, k) + P_x(i, j, k) \]

\[ E_y^{n+1}(i, j, k) \approx K_y(i, j, k)E_y^n(i, j, k) + P_y(i, j, k) \]

\[ E_z^{n+1}(i, j, k) \approx K_z(i, j, k)E_z^n(i, j, k) + P_z(i, j, k) \]

where \( K_x(i, j, k) = \frac{\varepsilon_x(i, j, k) - 0.5\Delta t\sigma_x(i, j, k)}{\varepsilon_x(i, j, k) + 0.5\Delta t\sigma_x(i, j, k)} \), and \( P_x(i, j, k) = \frac{\Delta t}{\varepsilon_x(i, j, k) + 0.5\Delta t\sigma_x(i, j, k)} \). Similarly for \( y \) and \( z \).
Finite Difference Time Domain Method (7)

The Gaussian pulse is a good waveform for computing the time domain response of a target. Practically speaking, a Gaussian pulse cannot be transmitted because DC does not radiate. The antenna will attenuate frequencies near zero and the radiated frequency spectrum will not be that of a Gaussian spectrum.
Finite Difference Time Domain Method (8)

Solution process:

1. Compute the scattered fields $\vec{E}_s(t), \vec{H}_s(t)$ throughout the grid by time stepping (also called “marching in time”). Store the fields over the grid boundary for a sufficient observation time $T_{\text{max}}$.

2. Fourier transform the scattered field on the boundary to get the frequency domain fields, $\tilde{E}_s(\omega), \tilde{H}_s(\omega)$.

3. Find the equivalent electric and magnetic surface currents on the grid boundary from the scattered fields: $\vec{J}_{s_{\text{eq}}}$ and $\vec{J}_{m_{\text{eq}}}$.

4. Apply the radiation integrals to get the far scattered fields and then compute the RCS. For two-dimensional problems (solution independent of $z$):

$$E_\phi(\rho, \phi) = -j\omega\mu A_\phi - jkF_z$$ and $$E_z(\rho, \phi) = -j\omega\mu A_z - jkF_\phi$$

where

$$\tilde{A}(\rho, \phi) = \frac{e^{-jkr}}{\sqrt{8jk\pi\rho}} \int_{S} \vec{J}_{s_{\text{eq}}} (\rho') e^{-jk\rho'\cos(\phi - \phi')} ds'$$

$$\tilde{F}(\rho, \phi) = \frac{e^{-jkr}}{\sqrt{8jk\pi\rho}} \int_{S} \vec{J}_{m_{\text{eq}}} (\rho') e^{-jk\rho'\cos(\phi - \phi')} ds'$$
One-, Two- and Three-Dimensional Cross Sections

• **One-dimensional problems:** The geometry varies in only one dimension. Example: a sheet or panel that extends infinitely in the $x$ and $y$ coordinates. The plane wave (Fresnel) reflection coefficient ($\Gamma$) is used for one-dimensional scattering:

$$\sigma_{1D} = |\Gamma|^2 \text{ (dimensionless*)}$$

• **Two-dimensional problems:** The geometry is independent of one coordinate. Example: an infinitely long cylinder with constant cross section. The echo width is used for two dimensional scattering:

$$\sigma_{2D} = \lim_{\rho \to \infty} 2\pi \rho \frac{|\vec{E}_s|^2}{|\vec{E}_t|^2} \text{ (meters)}$$

where $\rho$ is the radial variable.

• **Three-dimensional problems:** Conventional RCS is used for three dimensional scattering.

• Approximate conversion from echo width to radar cross section:

$$\sigma_{3D} = \sigma_{2D} \frac{L^2}{\lambda}$$

---

* In radar clutter models for extended surfaces, a parameter $\sigma^o$ with units of $m^2/m^2$ is used. This is only useful when a finite area of the surface $A_c$ is illuminated so that $\sigma = A_c \sigma^o$. 

---

* Version 4 (Nov. 2003)
Finite Difference Time Domain Method (9)

Calculation parameters:

1. Maximum frequency of computed RCS, $f_{\text{max}}$
2. Time step, $\Delta t$ (depends on $f_{\text{max}}$, $\mu$ and $\varepsilon$)
3. Observation time, $T_{\text{max}} = N_t \Delta t$
4. $N_t$ time steps (select $N_t = 2^M$, $M$ an integer, since $N_t$ is also the number of time samples for the FFT)
5. Frequency resolution, $\Delta f = \frac{1}{T_{\text{max}}}$
6. Gaussian pulse parameters: Effective duration, $T_{\text{eff}}$
   Effective bandwidth, $B_{\text{eff}}$
   Dynamic range, $R_{\text{dyn}}$
   Time duration of truncated Gaussian, $T_o$
7. Signal-to-noise ratio (“grid noise”)

\[
\text{SNR} = \frac{\text{POWER IN FREQUENCIES BELOW } f_{\text{max}}}{\text{POWER IN FREQUENCIES ABOVE } f_{\text{max}}}
\]
Finite Difference Time Domain Method (10)

Example: Scattering from an infinitely long square cylinder

Calculation parameters:

PEC scatterer
$L = 1.697 \text{ m}$
$\Delta \ell = 0.1061 \text{ m (16 cells)}$
$TM_z$ polarization
$SNR = 90 \text{ dB}$
$R_{\text{dyn}} = 120 \text{ dB}$
$B_{\text{eff}} = 1 \text{ GHz}$
$T_{\text{eff}} = 1/(2B_{\text{eff}}) = 0.5 \text{ ns}$
$f_{\text{max}} = B_{\text{eff}} = 1 \text{ GHz}$
$\Delta f = 10 \text{ MHz}$
$T_{\text{max}} = 1/\Delta f = 0.1 \mu s$
$N_t = \frac{T_{\text{max}}}{\Delta t} = 400$

$\rightarrow \text{ use } 2^M = 512$
Finite Difference Time Domain Method (11)

Time “snapshots” of the field in the computational grid
Finite Difference Time Domain Method (12)

Normalized far field scattering pattern at 550 MHz ($kL/2 \approx 10$). Note that the forward scatter is larger than the back scatter.
Summary

- FDTD is a time domain method based on the differential form of Maxwell’s equations.
- Convenient for very broadband excitations; i.e., the Fourier transform of time sampled scattered waveform provides the RCS over a wide bandwidth.
- A PML or ABC must be used at the grid boundary; large grids can be used so that the reflection from the grid boundary occurs very late in time and can be filtered out.
- Adaptable to arbitrary shapes and materials.
- Courant condition required for stability.
- No matrix manipulations necessary.
- Waveform choices: CW, ramped CW, Gaussian, rectangular and cosine pulses (Gaussian most common).

Problems:
1. Staircase errors: due to the target volume being represented by parallelepipeds; conformal techniques have been developed but are not easily incorporated.
2. Grid dispersion: numerical error in converting derivatives to finite differences causes the wave to move slower than its true velocity of propagation (frequency dependent).
3. Convergence: solution is converged when the field values in all of the cells have decayed to a sufficiently small value; i.e., wave has passed by and ringing has stopped.
The finite integration technique (FIT) is applied in the time domain, and is based on the integral form of Maxwell’s equations. The target and computational domain is discretized into subdomains, as in the case of the FDTD method. For illustrative purposes “bricks” are used as the cell geometry. The coordinates of a node \((x_i, y_j, z_k)\) is denoted by \((i, j, k)\).

Faraday’s law is \(\oint_{L} \vec{E}(\vec{r}, t) \cdot d\vec{l} = -\iint_{S} \frac{\partial}{\partial t} \vec{B}(\vec{r}, t) \cdot d\vec{s}\). For the front face, let:

\[
\begin{align*}
  e_x(i, j, k) &= \left[ \vec{E}(\vec{r}, t) \cdot d\vec{l} \right]_L \\
  b_z(i, j, k) &= \left[ \vec{B}(\vec{r}, t) \cdot d\vec{s} \right]_S 
\end{align*}
\]

\(L\) denotes the path around the perimeter of the face’s surface \(S\). Similarly, for other components of \(e\) and \(b\). Thus, Faraday’s law applied to the front face becomes

\[
e_x(i, j, k) - e_y(i+1, j, k) - e_x(i, j+1, k) - e_y(i, j, k) = -\frac{\partial}{\partial t} b_z(i, j, k)
\]
Finite Integration Technique (2)

Extend this to other faces, and then to all cells in the computational volume. Assemble the equations into a column vector with the degrees of freedom first in x, then y, and finally z. Therefore vectors of the form

\[ \vec{e} = [\vec{e}_x \mid \vec{e}_y \mid \vec{e}_z] \quad \text{and} \quad \vec{b} = [\vec{b}_x \mid \vec{b}_y \mid \vec{b}_z] \]

can be used to express the discretized Faraday’s Law as

\[
\begin{bmatrix}
\cdots \\
1 & 1 & \cdots & -1 & \cdots & -1 \\
\cdots \\
\end{bmatrix} \vec{e} = -\frac{\partial}{\partial t} \vec{b}
\]

\[ \overline{C} \]

represents the discrete curl operator with elements consisting of \{-1,0,1\}.

A discrete divergence operator can be generated starting from \( \iint_S \vec{B}(\vec{r}, t) \cdot d\vec{s} = 0 \). Using the previously defined quantities \( b \), the sum of the magnetic flux out of a cell must be zero:

\[-b_x(i, j, k) + b_x(i + 1, j, k) - b_y(i, j, k) + b_y(i, j + 1, k) - b_z(i, j, k) + b_z(i, j, k + 1) = 0\]
Finite Integration Technique (3)

Assemble the matrix over the entire cell complex, which yields the discrete divergence matrix

\[
\begin{pmatrix}
\cdots \\
-1 \cdots 1 \cdots -1 \cdots 1 \cdots -1 \cdots -1 \\
\cdots \\
\end{pmatrix}
\]

\[
\bar{b} = 0
\]

The discretization of the remaining two Maxwell’s equations (Ampere’s Law and Gauss’s Law) requires the introduction of a dual cell complex that is shifted \( \frac{1}{2} \) in all dimensions as shown. \( \tilde{G} \) denotes the dual grid.

**Ampere’s Law:**

\[
\oint_{\tilde{L}} \tilde{H}(\tilde{r}, t) \cdot d\ell = \iint_{\tilde{S}} \left[ \frac{\partial}{\partial t} \tilde{D}(\tilde{r}, t) + \tilde{J}(\tilde{r}, t) \right] \cdot ds
\]

**Gauss’s Law:**

\[
\iint_{S} \tilde{D}(\tilde{r}, t) \cdot ds = Q(\tilde{r}, t)
\]
Finite Integration Technique (4)

The procedure is applied to all of Maxwell’s equations, yielding the set

\[
\begin{align*}
\bar{C} \, \bar{e} &= -\frac{\partial}{\partial t} \, \bar{b} \\
\bar{S} \, \bar{b} &= 0 \\
\bar{C} \, \bar{h} &= -\frac{\partial}{\partial t} \, \bar{d} + \bar{j} \\
\bar{S} \, \bar{d} &= \bar{q}
\end{align*}
\]

and then continued with the constitutive relationships, potentials and continuity equation. For example, \( \vec{E} = -\nabla \Phi \) becomes

\[
\bar{e} = -\bar{C} \, \bar{\Phi}
\]

The fact that the discrete gradient matrix is the negative transpose of the dual discrete divergence operator has been used (\( \bar{C} = -\bar{S}^T \)).

The resulting set of equations uniquely and completely describe the fields and currents in a cell. They are presented here in a time-continuous space-discrete form, but are easily discretized in time (i.e., “marching in time”).
Summary of the FIT:

- FIT is a discretization method that transforms Maxwell’s equations in integral form onto a dual grid complex.
- The vectors of this scheme represent physically measurable quantities.
- The discretized equations result in sparse integer matrices \((\overline{C}, \overline{C'}, \overline{S}, \overline{S'})\) which only contain information on the incidence relations of the dual cell complex.
- Energy and charge conservation are easily checked, which can be used to evaluate convergence.
- The space and time stability (i.e., conservation of charge and energy) assure stable accurate long term calculations.
- Absorbing or radiation boundary conditions must be applied at the boundaries of the computational grid, just as in FDTD and FEM.
Microwave Studio (1)

- CST Microwave Studio uses the FIT
- Example: 3 wavelength PEC plate (dark green)
- Radiation boundaries (box outline)
- Plane wave normally incident ($x$ polarized)
- Close up of mesh
Microwave Studio (2)

$\phi = 0^\circ$

- Frequency: 300 MHz
- Main lobe magnitude: 38.9 dBsm
- Main lobe direction: 180.0 deg.
- Angular width (3 dB): 10.1 deg.
- Side lobe suppression: -0.0 dB

$\phi = 90^\circ$

- Frequency: 300 MHz
- Main lobe magnitude: 38.9 dBsm
- Main lobe direction: 180.0 deg.
- Angular width (3 dB): 10.1 deg.
- Side lobe suppression: -0.0 dB
Aircraft models in Microwave Studio

PEC except for canopy and nose cone

Composite wings with rounded leading edges
Microwave Studio (3)

Comparison of RCS at 600 MHz

PEC

Composite with rounded edges