Chapter 8  
NEAR-TO-FAR-FIELD TRANSFORMATION

CHAPTER OBJECTIVES:
• Understand the surface equivalent theorem for applications for radiation and scattering problems;
• Know how to calculate the frequency domain far-zone fields from the FDTD computed near field information;
• Know how to calculate the time domain far-zone fields from the FDTD computed near field information.

8.0  INTRODUCTION
The Finite Difference Time Domain (FDTD) has been used widely and extensively for the analysis and design of scattering and antenna radiation problems, which involve the manipulation of far-field information. We have formidable difficulties to analyze far-field problems based on all the techniques we have discussed in previous chapters. It seems that the far-field data can only be obtained by extending the FDTD lattice to the far field area, which is virtually impossible since the computing time and memory consumption will explode to unimaginable extent as the structure is physically enlarged. But the scattered fields in the zoned FDTD space, as described in section 3.2, are well preserved for a systematic near-to-far-field transformation, which is discussed in this chapter. Using all the scattered near-field information accumulated in a single FDTD modeling iteration, the near-to-far-field transformation can, efficiently and accurately, calculates the complete far-field response of a scattering structure or radiation pattern of an antenna. Therefore, it is not necessary to extend the FDTD lattice to collect far-field data.

The transformation technique is based on equivalence theorem and Green’s theorem, whose brief description will be given in following sections, and the numerical implementation equations of the transformation in Cartesian coordinates are derived below as well. The discussion of the frequency-domain near-to-far-field transformation concludes the chapter.

8.1 SURFACE EQUIVALENT THEOREM: HUYGEN’S PRINCIPLE
The surface equivalent theorem is a principle that can be used to replace actual sources, such as an antenna and transmitter. The fictitious sources are equivalent within a region because they produce within that region the same fields as the actual sources.

The surface equivalence theorem was introduced in 1936 by Schekunoff, and it is a more rigorous formulation of Huygen’s principle, which sates that
Each point on a primary wavefront can be considered to be a new source of secondary spherical wave and that a secondary wavefront can be constructed as the envelope of these secondary waves.

Based on the uniqueness theorem, if the tangential electric and magnetic fields are completely known over a closed surface, the fields in the source-free region can be uniquely determined.

**Original Problem**

\[
\left( \vec{E}_1, \vec{H}_1 \right) \\
(\mu_1, \varepsilon_1)
\]

---

**Equivalent Problem**

Fig. 8.1 Actual problem model.
Thus on the imaginary surface $S$ there must exist the equivalent currents:

$$
\mathbf{J}_s = \hat{n} \times (\mathbf{H}_1 - \mathbf{H})
$$

$$
\mathbf{M}_s = -\hat{n} \times (\mathbf{E}_1 - \mathbf{E})
$$

(8.1a)  
(8.1b)

Above equivalents are only equivalent to $V_2$, because they will produce the original fields $(\mathbf{E}_1, \mathbf{H}_1)$ on outside $S$. Different fields $(\mathbf{E}, \mathbf{H})$ will result in $V_1$. According to uniqueness theorem, we only need to know magnetic currents (tangential $\mathbf{E}$) or electric currents (tangential $\mathbf{H}$).

Since the fields within $S$, which is not the region of interest, can be anything. We assume that they are zero in the region. Thus

$$
\mathbf{J}_s = \hat{n} \times \mathbf{H}_1
$$

$$
\mathbf{M}_s = -\hat{n} \times \mathbf{E}_1
$$

(8.2a)  
(8.2b)

This form of field equivalent principle is known as Love’s equivalent principle.
We can actually fill $V_1$ with electric conductor or magnetic conductor as shown in Figs. 3.6 and 3.7. However, since this time the entire region becomes inhomogeneous, we must solve the radiating problem in the presence of the electric or magnetic conductor.

Fig. 8.3 Love’s equivalent principle model.

Fig. 8.4 Perfect electric conductor (PEC) equivalent.
Procedure of application of the surface equivalent principle:

- Select an imaginary surface that encloses the actual sources.
- Over the imaginary surface construct equivalent current densities \( \mathbf{J}_s, \mathbf{M}_s \) by using one of the following forms.
  
  a. \( \mathbf{J}_s, \mathbf{M}_s \) over \( S \) assuming that the \( (\mathbf{E}, \mathbf{H}) \) fields within \( S \) are not zero;
  
  b. \( \mathbf{J}_s, \mathbf{M}_s \) over \( S \) assuming that the \( (\mathbf{E}, \mathbf{H}) \) fields within \( S \) are zero (Love’s theorem);
  
  c. \( \mathbf{J}_s = 0, \mathbf{M}_s \) over \( S \) assuming that within \( S \) the medium is PEC;
  
  d. \( \mathbf{J}_s, \mathbf{M}_s = 0 \) over \( S \) assuming that within \( S \) the medium is PMC.

- Solve the equivalent problem.

**EXAMPLE**

A waveguide aperture is mounted on an infinite ground plane as shown Fig. 8.6. Assume that the tangential components of the electric fields are known as \( \mathbf{E}_a \). Find equivalent problem that will yield the same fields \( (\mathbf{E}, \mathbf{H}) \) radiated by the aperture to the right side of the interface.
Solution.

1) Choose the right side half space as the equivalent space and the left half space as the source region, and apply Love’ equivalent principle;

2) Apply PEC equivalent. Thus $\vec{J}_s = 0$ on the interface and $\vec{M}_s \neq 0$ on the aperture only;

3) Use the image theory for $\vec{M}_s$ on the PEC interface;

4) Double $\vec{M}_s$ and remove the PEC wall.

When the aperture $\vec{M}_s$ is found, we can further solve the radiated electromagnetic fields.

![Diagram of equivalent models for a waveguide aperture mounted on an infinite flat electric ground plane.](image)

Fig. 8.6 Equivalent models for a waveguide aperture mounted on an infinite flat electric ground plane.

Fig. 8.7 depicts schematically the most general case dealing with an arbitrarily three dimensional structure. We assume that a field $(\vec{E}_1, \vec{H}_1)$ filling all of space is generated by the actions of physical electric and magnetic current sources $(\vec{J}_1, \vec{M}_1)$ following in the surface of the structure of interest. In Fig 8.8b, we assume that $(\vec{J}_1, \vec{M}_1)$ are removed and
that there now exists a new field $(\vec{E}, \vec{H})$ inside an arbitrarily closed observation surface $S$ that completely enclose structure. For this desired situation to satisfy the required field boundary conditions on the tangential $E$- and $H$-field components at $S$, we must have the following nonphysical boundary conditions:

$$\begin{align*}
\vec{J}_s &= \hat{n} \times (\vec{H}_1 - \vec{H}) \\
\vec{M}_s &= -\hat{n} \times (\vec{E}_1 - \vec{E})
\end{align*}$$

(8.3a) (8.3b)

The virtual electric and magnetic currents of (8.3) radiate into free-space everywhere (inside and outside of $S$), and generate the original fields $(\vec{E}_1, \vec{H}_1)$ in the unbounded free-space outside $S$. Since the fields within $S$ can be anything, it is useful to assume that $(\vec{E}, \vec{H})$ inside $S$ are identically zero. Then the equivalent problem is reduced to

$$\begin{align*}
\vec{J}_s &= \hat{n} \times \vec{H}_1 \\
\vec{M}_s &= -\hat{n} \times \vec{E}_1
\end{align*}$$

(8.4a) (8.4b)

(a) Original interaction geometry.

(b) intermediate equivalent problem.
8.2 SOLUTION TO RADIATED FIELDS

It is common practice in the analysis of electromagnetic boundary-value problems to use auxiliary vector potentials as aids to obtaining solutions for the electric \( \vec{E} \) and magnetic \( \vec{H} \) fields. Most commonly used vector potential functions are the \( \vec{A} \), magnetic vector potential, and \( \vec{F} \), electric vector potential.

A two-step procedure is usually employed. First the vector potentials \( \vec{A} \) and \( \vec{F} \) can be found, once the boundary and source information are given. This is done by using the Green’s function integrating on arbitrary surface enclosing the source, which can be expressed as

\[
\vec{A} = \frac{\mu}{4\pi} \iiint_S \vec{J}(x',y',z') \frac{e^{-jBR}}{R} \, dv'
\]

(8.5a)

\[
\vec{F} = \frac{\varepsilon}{4\pi} \iiint_S \vec{M}(x',y',z') \frac{e^{-jBR}}{R} \, dv'
\]

(8.5b)

If \( \vec{J} \) and \( \vec{M} \) represent linear densities, (4.1) and (4.2) reduce to surface integrals

\[
\vec{A} = \frac{\mu}{4\pi} \iint_S \vec{J}_s(x',y',z') \frac{e^{-jBR}}{R} \, ds'
\]

(8.6a)

\[
\vec{F} = \frac{\varepsilon}{4\pi} \iint_S \vec{M}_s(x',y',z') \frac{e^{-jBR}}{R} \, ds'
\]

(8.6b)
If the observations are made in the far field \( (\beta R \gg 1) \), following approximation are valid

\[
R = \begin{cases} 
  r - r' \cos \Psi & \text{for phase variation} \\
  r & \text{for amplitude variation}
\end{cases} \quad \text{(8.7)}
\]

where \( \Psi \) is the angle between the source vector and observation point vector. Then (8.6) further reduces to

\[
\tilde{A} = \frac{\mathcal{M}}{4\pi} \iint_S \mathcal{J}_s(x',y',z') e^{-j\beta R} R ds' \cong \frac{\mathcal{M} e^{-j\beta R}}{4\pi R} \tilde{N}
\]

\[
\tilde{F} = \frac{\mathcal{E}}{4\pi} \iint_S \mathcal{J}_s(x',y',z') e^{-j\beta R} R ds' \cong \frac{\mathcal{E} e^{-j\beta R}}{4\pi R} \tilde{L}
\]

where

\[
\tilde{N} = \iint_S \mathcal{J}_s(x',y',z') e^{j\beta R \cos \psi} ds'
\]

\[
\tilde{L} = \iint_S \mathcal{M}_s(x',y',z') e^{j\beta R \cos \psi} ds'
\]

In the second step, the electric and magnetic fields are found after the vector potentials are determined. We know that the \( \tilde{E} \) and \( \tilde{H} \) fields are given by

\[
\tilde{E} = -j\omega \left[ \tilde{A} + \frac{1}{\beta^2} \nabla \left( \nabla \cdot \tilde{A} \right) - \frac{1}{\varepsilon} \nabla \times \tilde{F} \right]
\]

\[
\tilde{H} = -j\omega \left[ \tilde{F} + \frac{1}{\beta^2} \nabla \left( \nabla \cdot \tilde{F} \right) - \frac{1}{\mu} \nabla \times \tilde{A} \right]
\]

where \( \beta^2 = \omega^2 \mu \varepsilon \) is the wave number of the wave in the medium. Expand \( \tilde{A} \) and \( \tilde{F} \) as functions of \( r, \theta \) and \( \phi \), again since observation point is located in far-zone region, by neglecting higher-order terms of \( 1/r^n \) \( (1/r^n = 0, n = 2, 3, \ldots) \), the radiated \( \tilde{E} \) and \( \tilde{H} \) have only \( \theta \) and \( \phi \) components. We have following simplified forms

\[
\tilde{E} = -j\omega \tilde{A} - \frac{1}{\varepsilon} \nabla \times \tilde{F} \quad \text{\( \theta \) and \( \phi \) components only} \quad \text{(8.13a)}
\]

\[
\tilde{H} = -j\omega \tilde{F} - \frac{1}{\mu} \nabla \times \tilde{A} \quad \text{\( \theta \) and \( \phi \) components only} \quad \text{(8.13b)}
\]

Now we can write the \( \tilde{E} \) - and \( \tilde{H} \) -field components in the far field as
\[ E_r \equiv 0 \quad (8.14) \]
\[ E_o \equiv -j\omega [\tilde{A}_o + \eta \tilde{F}_\phi] \quad (8.15) \]
\[ E_\phi \equiv -j\omega [\tilde{A}_\phi - \eta \tilde{F}_o] \quad (8.16) \]
\[ H_r \equiv 0 \quad (8.17) \]
\[ H_o \equiv -\frac{j\omega}{\eta} [\tilde{A}_o - \eta \tilde{F}_o] \quad (8.18) \]
\[ H_\phi \equiv +\frac{j\omega}{\eta} [\tilde{A}_\phi + \eta \tilde{F}_\phi] \quad (8.19) \]

where \( \eta = \sqrt{\frac{\mu}{\varepsilon}} \). Since in our case only free-space will be involved, then \( \eta \) can be replaced by \( \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \), which is the intrinsic impedance of vacuum space. Using \( \tilde{A}_o, \tilde{A}_\phi, \tilde{F}_o, \) and \( \tilde{F}_\phi \) from (8.8) through (8.10), that is

\[
\tilde{A}_o = \frac{4\pi e^{-j\beta r}}{4\pi r} \tilde{N}_o
\quad (8.20)
\]
\[
\tilde{A}_\phi = \frac{4\pi e^{-j\beta r}}{4\pi r} \tilde{N}_\phi
\quad (8.21)
\]
\[
\tilde{F}_o = \frac{\varepsilon e^{-j\beta r}}{4\pi r} \tilde{L}_o
\quad (8.22)
\]
\[
\tilde{F}_\phi = \frac{\varepsilon e^{-j\beta r}}{4\pi r} \tilde{L}_\phi
\quad (8.23)
\]

we can reduce (8.14) through (8.19) to

\[ E_r \equiv 0 \quad (8.24) \]
\[ E_o \equiv -j\beta e^{-j\beta r} \left[ \tilde{L}_o + \eta \tilde{N}_o \right] \quad (8.25) \]
\[ E_\phi \equiv +j\beta e^{-j\beta r} \left[ \tilde{L}_\phi - \eta \tilde{N}_\phi \right] \quad (8.26) \]
\[ H_r \equiv 0 \quad (8.27) \]
\[ E_o \equiv -\frac{j\beta e^{-j\beta r}}{\eta} \left[ \tilde{N}_\phi - \tilde{L}_o \right] \quad (8.28) \]
\[ E_\phi \equiv \frac{j\beta e^{-j\beta r}}{\eta} \left[ \tilde{N}_o + \tilde{L}_\phi \right] \quad (8.29) \]

For radiators or scatterer, it is usually more convenient to represent above integrations in the rectangular coordinates, then (8.9) and (8.10) can be expressed as
\[ \tilde{N} = \int_S \int_{S'} (\tilde{J}_x (x',y',z') e^{jβr'\cos ψ} \, dx') = \int_S \int_{S'} (\tilde{a}_x J_x + \tilde{a}_y J_y + \tilde{a}_z j) e^{jβr'\cos ψ} \, dx' \quad (8.30) \]

\[ \tilde{L} = \int_S \int_{S'} (\tilde{M}_x (x',y',z') e^{jβr'\cos ψ} \, dx') = \int_S \int_{S'} (\tilde{a}_x M_x + \tilde{a}_y M_y + \tilde{a}_z j = M_z) e^{jβr'\cos ψ} \, dx' \quad (8.31) \]

Using the rectangular-to-spherical component transformation

\[
\begin{bmatrix}
\tilde{a}_x \\
\tilde{a}_y \\
\tilde{a}_z \\
\end{bmatrix} = \begin{bmatrix}
\sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\
\sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\
\cos \theta & -\sin \theta & 0
\end{bmatrix} \begin{bmatrix}
\tilde{a}_r \\
\tilde{a}_θ \\
\tilde{a}_ψ \\
\end{bmatrix}
\quad (8.32)
\]

we can reduce (8.30) and (8.31) for the \( \theta \) and \( ψ \) components to

\[ \tilde{N}_θ = \int_S (\tilde{J}_x \cos \theta \cos ϕ + \tilde{J}_y \cos \theta \sin ϕ - \tilde{J}_z \sin \theta) e^{jβr'\cos ψ} \, dx' \quad (8.33) \]

\[ \tilde{N}_ψ = \int_S (\tilde{M}_x \cos \theta \cos ϕ + \tilde{M}_y \cos \theta \sin ϕ - \tilde{M}_z \sin \theta) e^{jβr'\cos ψ} \, dx' \quad (8.34) \]

\[ \tilde{L}_θ = \int_S (\tilde{M}_x \sin \phi + \tilde{M}_y \cos \phi) e^{jβr'\cos ψ} \, dx' \quad (8.36) \]

It can be concluded from the above procedure that once we have the information of near-field, i.e. \( \tilde{M} \) and \( \tilde{J} \), by plugging in them into (8.24) through (8.29), we can come out with all the field components in far-field region.

### 8.3 CONSTRUCTING VIRTUAL SURFACE

As discussed earlier, the surface equivalence theorem can be briefly stated as following.

Assume that in free-space environment sources \((\tilde{J}_i, \tilde{M}_i)\) generate fields \((\tilde{E}_0, \tilde{H}_0)\). The same sources generate fields \((\tilde{E}_\text{total}, \tilde{H}_\text{total})\) when a scattering object is present. If we define the difference between the fields \(\tilde{E}_\text{total}\) and \(\tilde{E}_0\), and \(\tilde{H}_\text{total}\) and \(\tilde{H}_0\) as the scattered fields \(\tilde{E}_\text{scat}\) and \(\tilde{H}_\text{scat}\), namely

\[ \tilde{E}_\text{scat} = \tilde{E}_\text{total} - \tilde{E}_0 \quad (8.37) \]

\[ \tilde{H}_\text{scat} = \tilde{H}_\text{total} - \tilde{H}_0 \quad (8.38) \]

They satisfy the relations below

\[ \nabla \times \tilde{E}_\text{scat} = -\tilde{M}_\text{eq} - jω\mu_0 \tilde{H}_\text{scat} \quad (8.39) \]

\[ \nabla \times \tilde{H}_\text{scat} = -\tilde{J}_\text{eq} - jω\varepsilon_0 \tilde{E}_\text{scat} \quad (8.40) \]
where $\vec{J}_{eq}$ and $\vec{M}_{eq}$ are respectively the equivalent electric and magnetic surface current densities on an arbitrary surface enclosing the source. Equation (8.39) and (8.40) hold in the source and the free-space. As can be seen from the equations above, for analyzing the scattered fields, it is not necessary for us to know the details about the source enclosed, as far as the equivalent currents are given, we have the solution.

Apply the surface equivalence theorem to the general case in Cartesian coordinate system, we can construct a FDTD setup illustrated in Fig. 8.8. Assuming the S is a rectangular box of side dimensions $2x_0, 2y_0, 2z_0$ (or $i_0 \leq i \leq i_n$, $j_0 \leq j \leq j_n$ and $k_0 \leq k \leq k_n$, if expressed in a grid index). The equivalent electric and magnetic current surface densities are collected on the virtual surface located in the scattered-field region. To obtain the actual current densities over the enclosed virtual surface, we have following equations

$$\vec{J}_s = \vec{n} \times \vec{H}_s$$  \hspace{1cm} (8.41)
$$\vec{M}_s = -\vec{n} \times \vec{E}_s$$  \hspace{1cm} (8.42)

where $\vec{n}$ is the unit vector normally outward to the virtual surface and $\vec{E}_s$ and $\vec{H}_s$ are the electric and magnetic fields over the virtual surface, respectively.

Fig. 8.8 FDTD set-up of collecting equivalent current on virtual surface in scattered-field region.
Fig. 8.9 An example of collecting equivalent surface current on the virtual surface along \( x \) direction.

According to the equations above, we can easily obtain the numerical forms of the equivalent current densities. Obviously there are six faces to be accounted for the integration process. Three pairs of the faces have common properties in kernel functions. Notice that in an FDTD lattice the magnetic fields are half-cell away from the reference point, the interpolated values of adjacent two cells are adopted for calculation. The derivation on all six walls of the virtual surface are similar, let us first consider the field components on the left and right walls of the virtual surface, namely the two faces of \( S \) located at \( x' = \pm x_0 \). As shown in Fig. 8.9, \( \vec{J}_y \), \( \vec{J}_z \), \( \vec{M}_y \) and \( \vec{M}_z \) are the tangential surface currents on both walls that need to be formulated. For the current components over the left wall of the virtual surface positioned at \( x = -x_0 \) (or numerically \( i = i_o \)), we have
\[ \bar{J}_y \left( i_0, j + \frac{1}{2}, k \right) \]
\[ = -\tilde{a}_x \times \hat{a}_y H_z \left( i_0, j + \frac{1}{2}, k \right) \]
\[ = \tilde{a}_y H_z \left( i_0, j + \frac{1}{2}, k \right) \]
\[ = \tilde{a}_y \left[ \frac{\Delta x \left( i_0 + \frac{1}{2} \right) H_z \left( i_0 - \frac{1}{2}, j + \frac{1}{2}, k \right) - \Delta x \left( i_0 - \frac{1}{2} \right) H_z \left( i_0 + \frac{1}{2}, j + \frac{1}{2}, k \right)}{\Delta x \left( i_0 + \frac{1}{2} \right) + \Delta x \left( i_0 - \frac{1}{2} \right)} \right] \]  \hspace{1cm} (8.43)
\[ \bar{J}_z \left( i_0, j, k + \frac{1}{2} \right) \]
\[ = -\tilde{a}_x \times \hat{a}_y H_y \left( i_0, j, k + \frac{1}{2} \right) \]
\[ = \tilde{a}_z H_y \left( i_0, j, k + \frac{1}{2} \right) \]
\[ = \tilde{a}_z \left[ \frac{\Delta x \left( i_0 + \frac{1}{2} \right) H_y \left( i_0 - \frac{1}{2}, j, k + \frac{1}{2} \right) - \Delta x \left( i_0 - \frac{1}{2} \right) H_y \left( i_0 + \frac{1}{2}, j, k + \frac{1}{2} \right)}{\Delta x \left( i_0 + \frac{1}{2} \right) + \Delta x \left( i_0 - \frac{1}{2} \right)} \right] \]  \hspace{1cm} (8.44)

In the equations above, we have expressed \( H_y \left( i_0, j + \frac{1}{2}, k \right) \) and \( H_y \left( i_0, j, k + \frac{1}{2} \right) \) by interpolating the magnetic field components in adjacent cells correspondingly. Apparently for the uniform FDTD algorithm, in which \( \Delta x \left( i_0 + \frac{1}{2} \right) = \Delta x \left( i_0 - \frac{1}{2} \right) = \Delta x \) is valid, the two equations above reduce to

\[ \bar{J}_y \left( i_0, j + \frac{1}{2}, k \right) = \tilde{a}_y \left[ \frac{H_z \left( i_0 - \frac{1}{2}, j + \frac{1}{2}, k \right) - H_z \left( i_0 + \frac{1}{2}, j + \frac{1}{2}, k \right)}{2} \right] \]  \hspace{1cm} (8.45)
\[ \bar{J}_z \left( i_0, j, k + \frac{1}{2} \right) = \tilde{a}_z \left[ \frac{H_y \left( i_0 - \frac{1}{2}, j, k + \frac{1}{2} \right) - H_y \left( i_0 + \frac{1}{2}, j, k + \frac{1}{2} \right)}{2} \right] \]  \hspace{1cm} (8.46)

For the magnetic current densities, we have
The continuous integration in (8.33) through (8.36) can be discretized to summation when implemented with NU-FDTD algorithm. For the exponential phase terms one can obtain

\[ r' \cos \psi = \bar{r}' = (-x_0 \cdot \bar{x} + y' \cdot \bar{y} + z' \cdot \bar{z}) \cdot (\bar{x} \cdot \sin \theta \cos \phi + \bar{y} \cdot \sin \theta \sin \phi + \bar{z} \cdot \cos \phi) \]

\[ = -x_0 \cdot \sin \theta \cos \phi + y' \sin \theta \sin \phi + z' \cdot \cos \phi \]

The limits for the integrations are \(-y_0 \leq y' \leq y_n\) and \(-z_0 \leq z' \leq z_n\), or numerically expressed as \(j_0 \leq j' \leq j_n\) and \(k_0 \leq k' \leq k_n\). The differential integration area \(ds'\) equals to \(dy'dz'\). After all these substitutions, we can re-write (8.33) through (8.36) as below.

\[
\tilde{M}_y \left( i_0, j, k + \frac{1}{2} \right) = -(-\bar{a}_x) \times \bar{a}_z E_z \left( i_0, j, k + \frac{1}{2} \right) = -\bar{a}_y E_y \left( i_0, j, k + \frac{1}{2} \right) \tag{8.47}
\]

\[
\tilde{M}_y \left( i_0, j + \frac{1}{2}, k \right) = -(-\bar{a}_x) \times \bar{a}_z E_y \left( i_0, j + \frac{1}{2}, k \right) = -\bar{a}_z E_y \left( i_0, j + \frac{1}{2}, k \right) \tag{8.48}
\]
\[ \bar{L}_\phi = \sum_{j=\mu_j}^{\lambda_j} \sum_{k=\mu_k}^{\lambda_k} \left[ M_j \left( i_0, j + \frac{1}{2}, k \right) \cos \phi \right. \\
\left. \cdot \cos \left[ \beta \left( -x_0 \sin \theta \cos \phi + y' \sin \theta \sin \phi + z' \cos \theta \right) \right] \right] dy' dz' \]  

(8.53)

Similarly on the right wall of the virtual surface positioned at \( x = x_0 \) (or numerically \( i = i_n \)), we have

\[ \tilde{J}_y \left( i_n, j + \frac{1}{2}, k \right) = \tilde{a}_x \times \tilde{a}_z H_z \left( i_n, j + \frac{1}{2}, k \right) \]

\[ = -\tilde{a}_y H_z \left( i_n, j + \frac{1}{2}, k \right) \]

(8.54)

\[ \tilde{J}_z \left( i_n, j, k + \frac{1}{2} \right) = \tilde{a}_x \times \tilde{a}_y H_y \left( i_n, j, k + \frac{1}{2} \right) \]

\[ = -\tilde{a}_z H_y \left( i_n, j, k + \frac{1}{2} \right) \]

(8.55)

\[ \tilde{M}_y \left( i_n, j, k + \frac{1}{2} \right) = -\tilde{a}_x \times \tilde{a}_y E_y \left( i_n, j, k + \frac{1}{2} \right) = \tilde{a}_y E_y \left( i_n, j, k + \frac{1}{2} \right) \]

(8.56)

Only the exponential phase terms in the integration in (8.33) through (8.36) have to be adjusted as
\[ r' \cos \psi = \vec{r}' \cdot \hat{r} \]
\[ = (x_0 \cdot \hat{x} + y \cdot \hat{y} + z \cdot \hat{z}) \cdot (\hat{x} \cdot \sin \theta \cos \phi + \hat{y} \cdot \sin \theta \sin \phi + \hat{z} \cdot \cos \theta) \]
\[ = x_0 \cdot \sin \theta \cos \phi + y' \sin \theta \sin \phi + z' \cdot \cos \theta \]

while the other terms keep unchanged as on left wall of the virtual surface. We can obtain the equations on the other four walls by following the same procedure.

On the front wall:
\[
\vec{J}_x \left( i + \frac{1}{2}, j_0, k \right) \\
= -\vec{a}_y \times \hat{a}_z H_z \left( i + \frac{1}{2}, j_0, k \right) \\
= -\hat{a}_z H_z \left( i + \frac{1}{2}, j_0, k \right) \\
= -\hat{a}_z \left[ \Delta y (j_0 + \frac{1}{2}) H_z (i + \frac{1}{2}, j_0 - \frac{1}{2}, k) + \Delta y (j_0 - \frac{1}{2}) H_z (i + \frac{1}{2}, j_0 + \frac{1}{2}, k) \right] \\
\]

\[
\vec{J}_z \left( i, j_0, k + \frac{1}{2} \right) \\
= -\hat{a}_y \times \hat{a}_x H_x \left( i, j_0, k + \frac{1}{2} \right) \\
= \hat{a}_x H_x \left( i, j_0, k + \frac{1}{2} \right) \\
= \hat{a}_x \left[ \Delta y (j_0 + \frac{1}{2}) H_x (i, j_0 - \frac{1}{2}, k + \frac{1}{2}) + \Delta y (j_0 - \frac{1}{2}) H_x (i, j_0 + \frac{1}{2}, k + \frac{1}{2}) \right] \\
\]

\[
\vec{M}_x \left( i, j_0, k + \frac{1}{2} \right) = -(-\hat{a}_y) \times \hat{a}_z E_z \left( i, j_0, k + \frac{1}{2} \right) = \hat{a}_z E_z \left( i, j_0, k + \frac{1}{2} \right) \\
\vec{M}_z \left( i + \frac{1}{2}, j_0, k \right) = -(-\hat{a}_x) \times \hat{a}_y E_x \left( i + \frac{1}{2}, j_0, k \right) = -\hat{a}_y E_x \left( i + \frac{1}{2}, j_0, k \right) 
\]

On the rear wall:
\[ J_z \left( i + \frac{1}{2}, j_n, k \right) \]
\[ = \tilde{a}_y \times \tilde{a}_z H_z \left( i + \frac{1}{2}, j_n, k \right) \]
\[ = -\tilde{a}_x H_z \left( i + \frac{1}{2}, j_n, k \right) \]
\[ = \tilde{a}_x \left[ \Delta y \left( j_n + \frac{1}{2} \right) H_z \left( i + \frac{1}{2}, j_n - \frac{1}{2}, k \right) + \Delta y \left( j_n - \frac{1}{2} \right) H_z \left( i + \frac{1}{2}, j_n + \frac{1}{2}, k \right) \right] \]
\[ = -\tilde{a}_z H_x \left( i, j_n, k + \frac{1}{2} \right) \]
\[ = \tilde{a}_y \times \tilde{a}_x H_x \left( i, j_n, k + \frac{1}{2} \right) \]
\[ = -\tilde{a}_x H_x \left( i, j_n, k + \frac{1}{2} \right) \]
\[ = -\tilde{a}_z \left[ \Delta y \left( j_n + \frac{1}{2} \right) H_x \left( i, j_n - \frac{1}{2}, k + \frac{1}{2} \right) + \Delta y \left( j_n - \frac{1}{2} \right) H_x \left( i, j_n + \frac{1}{2}, k + \frac{1}{2} \right) \right] \]
\[ = -\tilde{a}_z \left( i, j_n, k + \frac{1}{2} \right) = \tilde{a}_y \times \tilde{a}_z E_z \left( i, j_n, k + \frac{1}{2} \right) = -\tilde{a}_z E_z \left( i, j_n, k + \frac{1}{2} \right) \]
\[ \tilde{M}_x \left( i, j_n, k + \frac{1}{2} \right) = -\tilde{a}_y \times \tilde{a}_x E_z \left( i + \frac{1}{2}, j_n, k \right) = \tilde{a}_x E_z \left( i + \frac{1}{2}, j_n, k \right) \]

For the exponential phase terms:

\[ r' \cos \psi = \vec{r}' \cdot \hat{r} \]
\[ = (x' \cdot \hat{x} + y_0 \cdot \hat{y} + z \cdot \hat{z}) \cdot (\hat{x} \cdot \sin \theta \cos \phi + \hat{y} \cdot \sin \theta \sin \phi + \hat{z} \cdot \cos \theta) \]
\[ = x' \cdot \sin \theta \cos \phi + y_0 \sin \theta \sin \phi + z' \cdot \cos \theta \]

where the positive sign applies for the front wall and the negative for the rear. The integration limits are \(-x_0 \leq x' \leq x_0\) and \(-z_0 \leq z' \leq z_0\) or numerically \(i_0 \leq i \leq i_n\) and \(k_0 \leq k \leq k_n\), and \(ds' = dx'dz'\).
\[
\bar{J}_x \left( i + \frac{1}{2}, j, k_0 \right) \\
= -\bar{a}_z \times \bar{a}_y H_y \left( i + \frac{1}{2}, j, k_0 \right) \\
= \bar{a}_x H_y \left( i + \frac{1}{2}, j, k_0 \right) \\
= \bar{a}_x H_y \left( i + \frac{1}{2}, j, k_0 \right) \\
\]
\[
\bar{J}_y \left( i, j + \frac{1}{2}, k_0 \right) \\
= -\bar{a}_z \times \bar{a}_x H_x \left( i, j + \frac{1}{2}, k_0 \right) \\
= -\bar{a}_x \left( i, j + \frac{1}{2}, k_0 \right) \\
= -\bar{a}_x \left( i, j + \frac{1}{2}, k_0 \right) \\
\]
\[
\bar{M}_x \left( i, j + \frac{1}{2}, k_0 \right) = -\left( -\bar{a}_z \right) \times \bar{a}_y E_y \left( i, j + \frac{1}{2}, k_0 \right) = \bar{a}_x E_x \left( i, j + \frac{1}{2}, k_0 \right) \\
\bar{M}_y \left( i + \frac{1}{2}, j, k_0 \right) = -\left( -\bar{a}_z \right) \times \bar{a}_x E_x \left( i + \frac{1}{2}, j, k_0 \right) = \bar{a}_y E_y \left( i + \frac{1}{2}, j, k_0 \right) \\
\]
And on the top wall:
\[
\bar{J}_x \left( i + \frac{1}{2}, j, k_n \right) \\
= \bar{a}_z \times \bar{a}_y H_y \left( i + \frac{1}{2}, j, k_n \right) \\
= \bar{a}_x H_y \left( i + \frac{1}{2}, j, k_n \right) \\
= \bar{a}_x H_y \left( i + \frac{1}{2}, j, k_n \right) \\
\]
\[
\bar{M}_x \left( i, j + \frac{1}{2}, k_n \right) = -\left( -\bar{a}_z \right) \times \bar{a}_y E_y \left( i, j + \frac{1}{2}, k_n \right) = \bar{a}_x E_x \left( i, j + \frac{1}{2}, k_n \right) \\
\bar{M}_y \left( i + \frac{1}{2}, j, k_n \right) = -\left( -\bar{a}_z \right) \times \bar{a}_x E_x \left( i + \frac{1}{2}, j, k_n \right) = \bar{a}_y E_y \left( i + \frac{1}{2}, j, k_n \right) \\
\]
\[ J_y(i, j + \frac{1}{2}, k_n) = \tilde{a}_z \times \tilde{a}_x H_x(i, j + \frac{1}{2}, k_n) \]
\[ = -\tilde{a}_y(i, j + \frac{1}{2}, k_n) \]  
\[ = \tilde{a}_y(i, j + \frac{1}{2}, k_n) \]  
(8.73)

\[ \tilde{M}_x(i, j + \frac{1}{2}, k_n) = -\tilde{a}_z \times \tilde{a}_y E_y(i, j + \frac{1}{2}, k_n) = \tilde{a}_x E_x(i, j + \frac{1}{2}, k_n) \]  
\[ \tilde{M}_y(i + \frac{1}{2}, j, k_n) = -\tilde{a}_z \times \tilde{a}_x E_x(i + \frac{1}{2}, j, k_n) = -\tilde{a}_y E_y(i + \frac{1}{2}, j, k_n) \]  
(8.74)

(8.75)

For the exponential phase terms:

\[ r' \cos \psi \]
\[ = \tilde{r} \cdot \tilde{r} \]
\[ = (x \cdot \tilde{x} + y' \cdot \tilde{y} \pm z_0 \cdot \tilde{z}) \cdot (\tilde{x} \cdot \sin \theta \cos \phi + \tilde{y} \cdot \sin \theta \sin \phi + \tilde{z} \cdot \cos \theta) \]
\[ = x \cdot \sin \theta \cos \phi + y' \sin \theta \sin \phi \pm z_0 \cdot \cos \theta \]  
(8.76)

where the positive sign applies for bottom wall and the negative for the top. The integration limits are \(-x_0 \leq x' \leq x_0\) and \(-y_0 \leq y' \leq y_0\) or numerically \(i_0 \leq i \leq i_n\) and \(j_0 \leq j \leq j_n\), and \(ds' = dx' dy'\).

### 8.4 Radar Cross Section of Scattering Structures

An important parameter in scattering problems is the electromagnetic scattering by a target which is generally represented by its radar cross section (RCS) \((\sigma)\). The bistatic RCS is defined as the area intercepting the amount of power that, when scattered isotropically, produces at the receiver a density that is equal to the density scattered by the actual target.

In equation form the RCS of a three-dimensional target can be written as

\[ RCS_{3-D}(\theta, \phi) = \lim_{r \to \infty} \left[ 4\pi r^2 \frac{P_s}{P^i} \right] \]  
(8.77)
\[ = \lim_{r \to \infty} 4\pi r^2 \left| \frac{\tilde{E}^s}{\tilde{E}^i} \right|^2 \]  
\text{(8.78)}

\[ = \lim_{r \to \infty} 4\pi r^2 \left| \frac{\tilde{H}^s}{\tilde{H}^i} \right|^2 \]  
\text{(8.79)}

where \( P^s \), \( P^i \) are scattered and incident power densities, respectively, \( \tilde{E}^s, \tilde{E}^i \) are scattered and incident electric fields, respectively and \( \tilde{H}^s, \tilde{H}^i \) are scattered and incident magnetic fields, respectively.

The scattered power densities can be achieved by averaging the Poynting vector over time as

\[ P^s = \frac{1}{2} \Re \left( \tilde{E}_\phi \cdot \tilde{H}_\phi^* \right) + \frac{1}{2} \Re \left( -\tilde{E}_\phi \cdot \tilde{H}_\phi^* \right) = \frac{\beta^2}{32\pi^2 \eta_0 r^2} \left( \left| \tilde{E}_\phi + \eta_0 \tilde{N}_\phi \right|^2 + \left| \tilde{L}_\phi - \eta_0 \tilde{N}_\phi \right|^2 \right) \]  
\text{(8.80)}

Assuming the plane wave incidence, then the incident power densities can be expressed as

\[ P^i = \frac{1}{2} \Re \left( \tilde{E}_i \cdot \tilde{H}_i^* \right) = \frac{E_0^2}{2\eta_0} \]  
\text{(8.81)}

where \( E_0 \) is the amplitude of the electric field of the incident plane wave. By substituting (8.80) and (8.81) into (8.77) we obtain the final expression for bistatic RCS

\[ RCS_{3-D}(\theta, \phi) = \frac{\beta^2}{4\pi E_0^2} \left( \left| \tilde{L}_\phi + \eta_0 \tilde{N}_\phi \right|^2 + \left| \tilde{L}_\phi - \eta_0 \tilde{N}_\phi \right|^2 \right) \]  
\text{(8.82)}

### 8.5 FREQUENCY-DOMAIN NEAR-TO-FAR-FIELD TRANSFORMATION

There are two alternatives performing near-to-far-field transformation in FDTD lattice:

- “on-the-fly” time-domain transformation that calculates the time waveforms of the scattered or radiated E- and H-fields at previously specified angular positions in the far field. These calculations are performed simultaneously with the normal FDTD time-stepping.

- frequency-domain transformation that computer transformation of the fields in whole far field region at a selected frequency point. First the Discrete Fourier Transform (DFT) has to be performed on the collected equivalent current
densities over virtual surface within the FDTD iteration loop. Then based on the results near-to-far-field transformation is conducted on the specified frequency point.

In this research the second method has been implemented in the solver.

First of all, let us review the DFT process and the implementation in the FDTD calculation. The Fourier Transform of a function $f$ is defined as

$$F(f)(\omega) = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

By numerically approximating the Fourier Transform we define the discrete Fourier transform $G$ of $f$ as

$$G\left(\frac{2n\pi}{NT}\right) = T\sum_{k=0}^{N-1} f(kT)e^{-j2\pi nk/N} \quad \text{for } n = 0,1,2,\cdots,N-1$$

where $T > 0$ (T is the sampling period) and $N$ is a positive integer. Using the FDTD notations $\Delta t$ for $T$ and let $\omega_0 = 2\pi f_0 = \frac{2n\pi}{N\Delta t}$, then (8.84) becomes

$$G(2\pi f_0) = G\left(\frac{2n\pi}{N\Delta t}\right) = \Delta t\sum_{k=0}^{N-1} f(k\Delta t)e^{-j2\pi f_0\Delta k}$$

in which $f_0$ is the frequency point of interest and $N$ is the total time step of the FDTD iteration.

Now the frequency-domain near-to-far-field approach is summarized based on all the discussion in last three sections. First, assuming that the frequency point of interest is defined as $f_0$. In the time-stepping of the FDTD iteration, the current densities collected on the virtual surface in each step will be transformed to frequency domain by applying DFT. Simply substitute function $f(k\Delta t)$ in (8.85) with $\tilde{J}$ or $\tilde{M}$

$$\tilde{J}(2\pi f_0) = \Delta t\sum_{k=0}^{N-1} \tilde{J}(k\Delta t)e^{-j2\pi f_0\Delta k} \quad (8.86)$$

$$\tilde{M}(2\pi f_0) = \Delta t\sum_{k=0}^{N-1} \tilde{M}(k\Delta t)e^{-j2\pi f_0\Delta k} \quad (8.87)$$

And after the FDTD iteration, the transformed $\tilde{J}$ and $\tilde{M}$ will be plugged into (8.33) through (8.36) for integration over the whole virtual surface. For radiation problems, most of the time radiation patterns on principal planes (azimuthal and horizontal planes) are expected. These can be obtained by using the formulation of (8.24) through (8.29)
For scattering problems, bistatic RCS can be achieved by performing calculation of (8.82).