A WEAK GALERKIN FINITE ELEMENT METHOD FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION-REACTION PROBLEMS

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Abstract. In this article, a new weak Galerkin finite element method is introduced to solve convection-diffusion-reaction equations in the convection dominated regime. Our method is highly flexible by allowing the use of discontinuous approximating functions on polytopal mesh without imposing extra conditions on the convection coefficient. Error estimate is devised in a suitable norm. Numerical examples are provided to confirm theoretical findings and efficiency of the method.

Key words. weak Galerkin, finite element method, discrete gradient, singular perturbation, convection-diffusion-reaction, polyhedral mesh

AMS subject classifications. Primary: 65N15, 65N30; Secondary: 35J50

1. Introduction. In this work, we consider a finite element method (FEM) for the following convection-diffusion-reaction problems exhibiting layer behavior

\begin{align}
-\epsilon \Delta u + \nabla \cdot (bu) + cu & = f \quad \text{in } \Omega \subset \mathbb{R}^d, \\
u & = 0 \quad \text{on } \partial \Omega,
\end{align}

where \( \epsilon > 0 \) is a parameter, \( d = 2 \) or 3, and \( \Omega \) is a polygonal (when \( d = 2 \)) or polyhedral (when \( d = 3 \)) domain with boundary \( \partial \Omega \). For well-posedness of the differential equation, we assume that \( b, c, \) and \( f \) are sufficiently smooth, \( b \in [W^{1,\infty}(\Omega)]^d \), and \( c + \frac{1}{2} \nabla \cdot b \geq c_0 > 0 \) for some constant \( c_0 \) (cf., e.g., [34]).

It is well-known that the linear elliptic equation (1.1)-(1.2), though looks simple, is very difficult to solve when it is singularly perturbed (i.e., when the singular perturbation parameter \( \epsilon \ll 1 \)). Due to the very small diffusion coefficient, the solution of the boundary value problem typically possesses layers, which are thin regions where the solution and/or its derivatives change rapidly. Standard numerical methods fail to provide accurate approximations in this case unless the computational mesh is of the magnitude of the layers.

Numerical stabilization techniques have been developed to resolve the difficulty, which can be roughly classified into fitted mesh methods and fitted operator methods. Efforts toward the optimization of numerical meshes can be traced back to the works of Bakhvalov [5], whose meshes are obtained from projections of equidistant partition of layer functions. An easier yet efficient idea of piecewise-equidistant meshes proposed by Shishkin [36] has attracted tremendous attentions and set a trend of studies of problems with singular perturbation, which is still in vogue. In addition, adaptive

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methods have been gaining popularity for more than four decades \[4\] (cf. also \[11\]), which have been addressed a variety of difficulties including layers \[32\]. For details of fitted mesh methods on singularly perturbed problems, we refer to the books \[22, 23, 34, 37\] and the references therein.

When solving the problems of complicated domains or layer structures, fitted operator methods, dominated by upwind-type schemes, are usually applied. The mechanism of upwinding is to add artificial diffusion/viscosity to balance the convection and to stabilize a standard discretization method. Upwind schemes were first proposed as finite difference methods, which were later extended to FEMs. Among variations of approaches, the streamline upwind/Petrov-Galerkin (SUPG) method introduced by Hughes and Brooks is one of the most successful methods \[19\]; see also \[7\]. The SUPG method improves stability of the standard Galerkin method by adding residuals with discontinuous weights. A drawback of upwind methods is that too much artificial diffusion will smear layers, which is a particular concern for the problems in multiple dimensions. For more discussions about upwind methods, please refer to \[34\] and its references.

In the last two decades, the flexibility and special capabilities of utilizing discontinuous discrete trial and test spaces have prompted vigorous developments of a variety of methods, including discontinuous Galerkin (DG) methods, weak Galerkin (WG) methods, etc. The DG method, also known as boundary penalty method and interior penalty method in different contexts, originated in the early 1970s; see, e.g., \[3, 14, 29, 31, 41\]. It has been a major technique for solving conservation laws, which later becomes a very popular approach for elliptic problems \[1\]. When applied to convection-diffusion problems, the DG approach includes a natural upwinding that is equivalent to some stabilization \[21, 34\]. Literature on DG methods for convection-diffusion problems includes \[2, 6, 8, 15, 18, 20\]. For detailed discussions on DG methods, we refer to \[12, 16, 33\]. The WG method is a recently developed novel framework for solving partial differential equations \[38\]. With new concepts referred as weak function and weak gradient, the numerical approach allows the use of totally discontinuous functions and has a simple formulation with parameter independent. The WG method has been rapidly developed and been applied to different types of problems, including second order elliptic problems, biharmonic problems, and Stokes flow, etc.; see, e.g., \[24, 25, 26, 27, 28, 38, 40\]. WG methods allow adoption of general polytopal meshes, which provide enormous flexibility in scheme design and implementation. Recently, a WG method is studied in \[9\] for problem (1.1)-(1.2) with a set of strict assumptions on data of the differential equation.

The goal of this article is to develop a new simple WG method for the convection-diffusion-reaction problem (1.1)-(1.2) with singular perturbation. The proposed method is also a fitted operator type method, which yields stable numerical solutions over uniform meshes. Our WG method, like the DG method, involves upwinding by using discontinuous approximations across elements but not artificial residuals, which hence requires more degrees of freedom than many other upwind-type schemes, such as the SUPG method.

Compared to many existing methods, our method has some unique features: 1) Different stabilization technique is used to avoid smearing the layers by over stabilization. 2) A set of strict assumptions for convection coefficient in \[9\] and other papers is removed. 3) Error analysis has been significantly simplified. Our error analysis is clean and short. 4) The discontinuity between elements makes it possible to use polytopal meshes (see Section 2), which allow remarkable convenience in adaptive
mesh generation and implementation; cf., e.g., [10, 13, 25, 26]. When shape-regular meshes are used, a priori error estimates of $O(h^{k+1/2})$ for $k$th order element can be obtained. Extensive numerical experiments have been conducted. Numerical results show that the proposed WG method is efficient for different types of singularly perturbed convection-diffusion-reaction problems.

Throughout this article, we use $C$ for generic constants independent of $\epsilon$, mesh size, and the solution to equation (1.1)-(1.2), which may not necessarily be the same at each occurrence.

The rest of this article is organized as follows. In Section 2, a new WG FEM is proposed. Analysis of the new method is found in Section 3. Numerical experiments are presented in Section 4 to support the theoretical results.

### 2. Weak Galerkin Finite Element Schemes

For any given polyhedron $D \subseteq \Omega$, we use the standard definition of Sobolev spaces $H^s(D)$ with $s \geq 0$. The associated inner product, norm, and semi-norms in $H^s(D)$ are denoted by $(\cdot, \cdot)_{s,D}$, $\| \cdot \|_{s,D}$, and $| \cdot |_{s,D}$, respectively. When $s = 0$, $H^0(D)$ coincides with the space of square integrable functions $L^2(D)$. In this case, the subscript $s$ is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript $D$ is also suppressed when $D = \Omega$.

![Fig. 2.1. Depiction of a shape-regular polygonal element ABCDEFA.](image)

Let $T_h$ be a partition of the domain $\Omega$ consisting of elements which are closed and simply connected polygons in two dimension or polyhedra in three dimension; see Fig. 2.1. Let $E_h$ be the set of all edges or flat faces in $T_h$, and $E_h^0 = E_h \setminus \partial \Omega$ be the set of all interior edges or flat faces. Denote by $h_T$ the diameter for every element $T \in T_h$ and $h = \max_{T \in T_h} h_T$ the mesh size for $T_h$. We need some shape regularity assumptions for the partition $T_h$ described as below (cf. [39]).

**A1:** Assume that there exist two positive constants $\varrho_v$ and $\varrho_e$ such that for every element $T \in T_h$ we have

$$
(2.1) \quad \varrho_v h_T^d \leq |T|, \quad \varrho_e h_e^{d-1} \leq |e|
$$

for all edges or flat faces of $T$. 
A2: Assume that there exists a positive constant $\kappa$ such that for every element $T \in \mathcal{T}_h$ we have
\begin{equation}
\kappa h_T \leq h_e
\end{equation}
for all edges or flat faces $e$ of $T$.

A3: Assume that the mesh edges or faces are flat. We further assume that for every $T \in \mathcal{T}_h$, and for every edge/face $e \in \partial T$, there exists a pyramid $P(e, T, A_e)$ contained in $T$ such that its base is identical with $e$, its apex is $A_e \in T$, and its height is proportional to $h_T$ with a proportionality constant $\sigma_e$ bounded away from a fixed positive number $\sigma^*$ from below. In other words, the height of the pyramid is given by $\sigma_e h_T$ such that $\sigma_e \geq \sigma^* > 0$. The pyramid is also assumed to stand up above the base $e$ in the sense that the angle between the vector $x_e - A_e$, for any $x_e \in e$, and the outward normal direction of $e$ is strictly acute by falling into an interval $[0, \theta_0]$ with $\theta_0 < \pi/2$.

A4: Assume that each $T \in \mathcal{T}_h$ has a circumscribed simplex $S(T)$ that is shape regular and has a diameter $h_{S(T)}$ proportional to the diameter of $T$; i.e., $h_{S(T)} \leq \gamma_s h_T$ with a constant $\gamma_s$ independent of $T$. Furthermore, assume that each circumscribed simplex $S(T)$ interests with only a fixed and small number of such simplices for all other elements $T \in \mathcal{T}_h$.

For a given integer $k \geq 1$, let $V_h$ be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows
\begin{equation}
V_h = \{ v = \{v_0, v_b\} : v_0|_T \in \mathbb{P}_k(T), \ v_b|_e \in \mathbb{P}_k(e), \ e \subset \partial T, \ T \in \mathcal{T}_h \}
\end{equation}
and
\begin{equation}
V^0_h = \{ v : v \in V_h, \ v_b = 0 \text{ on } \partial \Omega \},
\end{equation}
where $\mathbb{P}_k$ is the space of polynomials of total degree $k$ or less. We would like to emphasize that any function $v = \{v_0, v_b\} \in V_h$ has a single value $v_b$ on each edge $e \in \mathcal{E}_h$, which is not necessary to be the trace of $v_0$ on $e$. The discontinuity between elements allows us to apply $\mathbb{P}_k$ spaces on polytopal elements, which brings convenience in many aspects in implementation.

For any $v = \{v_0, v_b\}$, a weak gradient $\nabla_w v \in [\mathbb{P}_{k-1}(T)]^d$ is defined on $T$ as the unique polynomial satisfying
\begin{equation}
(\nabla_w v, \tau)_T = -(v_0, \nabla \tau)_T + \langle v_b, \tau \cdot \mathbf{n} \rangle_{\partial T} \quad \forall \tau \in [\mathbb{P}_{k-1}(T)]^d,
\end{equation}
where $\mathbf{n}$ is the unit outward normal vector to $\partial T$, and $(\cdot, \cdot)_T$ and $(\cdot, \cdot)_{\partial T}$ are standard $L^2$ inner products on $T$ and $\partial T$, respectively. For any $v = \{v_0, v_b\}$, we define a weak divergence $\nabla_w \cdot (\mathbf{b} v) \in \mathbb{P}_k(T)$ related to $\mathbf{b}$ on $T$ as the unique polynomial satisfying
\begin{equation}
(\nabla_w \cdot (\mathbf{b} v), w)_T = -(\mathbf{b} v_0, \nabla w)_T + \langle \mathbf{b} \cdot \mathbf{n} v_b, w \rangle_{\partial T} \quad \forall w \in \mathbb{P}_k(T).
\end{equation}

We introduce four global projections $Q_0$, $Q_b$, $Q_h$ and $Q_{h_b}$. They are element-wise defined $L^2$ projections detailed as follows. For each element $T \in \mathcal{T}_h$, $Q_0 : L^2(T) \to \mathbb{P}_k(T)$ and $Q_b : L^2(e) \to \mathbb{P}_k(e)$ are the $L^2$ projections onto the associated local polynomial spaces, $Q_b : [L^2(T)]^d \to [\mathbb{P}_{k-1}(T)]^d$ is the $L^2$ projection onto the local weak gradient space. Finally, we define a projection operator $Q_h u = \{Q_0 u, Q_b u\} \in V_h$ for the true solution $u$. 
For simplicity, we adopt the following notations,
\[
(v, w)_T = \sum_{T \in T_h} (v, w)_T = \sum_{T \in T_h} \int_T v w d\mathbf{x},
\]
\[
(v, w)_{\partial T} = \sum_{T \in T_h} (v, w)_{\partial T} = \sum_{T \in T_h} \int_{\partial T} v w d\mathbf{s}.
\]

For \( v = \{v_0, v_b\} \) and \( w = \{w_0, w_b\} \) in \( V_h \), we define a bilinear form as
\[
(2.7) \quad a(v, w) = \epsilon (\nabla w \cdot \nabla v)_{T_h} + (\nabla w \cdot (b v), w_0)_{T_h} + (c v_0, w_0) + s_c(v, w) + s_d(v, w),
\]
where
\[
s_c(v, w) = \sum_{T \in T_h} \langle b \cdot n (v_0 - v_b), w_0 - w_b \rangle_{\partial T},
\]
\[
s_d(v, w) = \sum_{T \in T_h} \epsilon h_T^{-1} (v_0 - v_b, w_0 - w_b)_{\partial T},
\]
and
\[
\partial_+ T = \{ x \in \partial T : b(x) \cdot n(x) \geq 0 \}.
\]

Then a WG FEM is proposed as the following.

**Algorithm 1 (Weak Galerkin Method).** A weak Galerkin approximation for (1.1)-(1.2) is to seek \( u_h = \{u_0, u_b\} \in V_h^0 \) satisfying the following equation:
\[
(2.8) \quad a(u_h, v) = (f, v_0) \quad \forall \ v = \{v_0, v_b\} \in V_h^0.
\]

Accordingly, we define an energy norm \( \| \cdot \| \) in \( V_h^0 \): for any \( v \in V_h^0 \),
\[
(2.9) \quad \| v \|^2 = \epsilon \sum_{T \in \mathcal{T}_h} \| \nabla v \|^2_T + \sum_{T \in \mathcal{T}_h} \| b \cdot n \|^{1/2} (v_0 - v_b) \|^2_{\partial T} + \| v_0 \|^2 + s_d(v, v).
\]

**Remark 1.** Compared to the WG scheme proposed in [9], our stabilizer for convection term \( s_c(\cdot, \cdot) \) is simple and has upwinding flavor. Moreover, unlike in [2, 9], our WG method does not make extra assumptions on the convection coefficient \( b \).

**3. Analysis.** In this section, we will establish the well-posedness of the WG method (2.8) and obtain error estimate for the weak Galerkin finite element approximation \( u_h \).

**3.1. Well-posedness of the WG method and error equation.** Verification of continuity of the bilinear form (2.7) is straightforward. We next show the coercivity of the bilinear form. The following lemma is useful.

**Lemma 3.1.** For \( v, w \in V_h^0 \), then
\[
(\nabla w \cdot (b v), w_0)_{T_h} = (\nabla \cdot b v_0, w_0)_{T_h} - (v_0, \nabla w \cdot (b w))_{T_h} - (b \cdot n (v_0 - v_b), w_0 - w_b)_{\partial T_h}.
\]
Proof. It follows from the definition of the weak divergence (2.6) that

\[
(\nabla_w \cdot (bv), w_0)_{\partial T_h} = -(bw_0, \nabla w_0)_{\partial T_h} + (b \cdot n w_0, w_0)_{\partial T_h}
\]

\[
= (\nabla \cdot bv_0, w_0)_{\partial T_h} + (bw_0, \nabla w_0)_{\partial T_h} - (b \cdot n(v_0 - v_b), w_0)_{\partial T_h}
\]

\[
= (\nabla \cdot bv_0, w_0)_{\partial T_h} - (\nabla_w \cdot (bw), v_0)_{\partial T_h} + (b \cdot n w_0, v_0)_{\partial T_h}
\]

which implies (3.1). Here we use the facts that \( v, w \in V_h^0 \) and \( (b \cdot n v, w_h)_{\partial T_h} = 0 \) to obtain the last equation above.

**Lemma 3.2.** For \( v \in V_h^0 \), then

\[
(3.2) \quad C \|v\|^2 \leq a(v, v).
\]

**Therefore, the WG formulation (2.8) has a unique solution.**

**Proof.** It follows from (3.1) that for any \( v \in V_h^0 \),

\[
(3.3) \quad (\nabla_w \cdot (bv), v_0)_{\partial T_h} = \frac{1}{2} (\nabla \cdot bv_0, v_0)_{\partial T_h} - \frac{1}{2} (b \cdot n(v_0 - v_b), v_0 - v_b)_{\partial T_h}.
\]

Using (3.3), we have

\[
a(v, v) = c(\nabla_w v, \nabla_w v)_{\partial T_h} + ((c + \frac{1}{2} \nabla \cdot b)v_0, v_0)_{\partial T_h}
\]

\[
- \frac{1}{2} (b \cdot n(v_0 - v_b), v_0 - v_b)_{\partial T_h} + s_c(v, v) + s_d(v, v)
\]

\[
\geq c(\nabla_w v, \nabla_w v)_{\partial T_h} + c_0 \|v_0\|^2 + \frac{1}{2} \sum_{T \in T_h} |||b \cdot n|||^{1/2}(v_0 - v_b)_{\partial T} + s_d(v, v)
\]

\[
\geq C \|v\|^2,
\]

which completes the proof. \( \square \)

We next derive an equation that the error satisfies, which will be used in error analysis below.

**Lemma 3.3.** Let \( u \) be the solution of the problem (1.1)-(1.2). Then for \( v \in V_h^0 \),

\[
(3.4) \quad (\nabla \cdot (bu), v_0) = (\nabla_w \cdot (bQ_hu), v_0)_{\partial T_h} - \ell_1(u, v) + \ell_2(u, v),
\]

where

\[
\ell_1(u, v) = (u - Q_0 u, b \cdot \nabla v_0)_{\partial T_h},
\]

\[
\ell_2(u, v) = (u - Q_k u, b \cdot n(v_0 - v_b))_{\partial T_h}
\]

**Proof.** It follows from the definition of the weak divergence (2.6) that

\[
(\nabla \cdot (bu), v_0)_{\partial T_h} = -(bu, \nabla v_0)_{\partial T_h} + (b \cdot nu, v_0)_{\partial T_h}
\]

\[
= -(bQ_0 u, \nabla v_0)_{\partial T_h} - \ell_1(u, v) + (b \cdot nQ_h u, v_0)_{\partial T_h}
\]

\[
- (b \cdot nQ_b u, v_0)_{\partial T_h} + (b \cdot nu, v_0 - v_b)_{\partial T_h}
\]

\[
= (\nabla_w \cdot (bQ_h u), v_0)_{\partial T_h} - \ell_1(u, v) + \ell_2(u, v).
\]
which implies (3.4). □

**Lemma 3.4.** Let $u$ be the solution of the problem (1.1)-(1.2). Then for $v \in V_h^0$,

\begin{equation}
-\epsilon(\Delta u, v_0) = \epsilon((\nabla w Q_h u, \nabla w v)_{\mathcal{T}_h} - \ell_u(u, v),
\end{equation}

where

\begin{equation}
\ell_u(u, v) = \epsilon((\nabla u - Q_h \nabla u) \cdot n, v_0 - v_b)_{\partial \mathcal{T}_h}.
\end{equation}

**Proof.** We refer the proof of the lemma to [26]. □

Let $\ell_b(u, v) = \ell_1(u, v) - \ell_2(u, v)$ and $\ell_c(u, v) = -(cu - cQ_0 u, v_0)$. We have the following error equation.

**Lemma 3.5.** Let $e_h = Q_h u - u_h \in V_h^0$. Then, for any $v \in V_h^0$ we have

\begin{equation}
a(e_h, v) = \ell_a(u, v) + \ell_b(u, v) + \ell_c(u, v) + s_c(Q_h u, v) + s_d(Q_h u, v).
\end{equation}

**Proof.** Testing (1.1) by $v = \{v_0, v_b\} \in V_h^0$, we arrive at

\begin{equation}
-\epsilon(\Delta u, v_0) + (\nabla \cdot (b u), v_0) + (cu, v_0) = (f, v_0).
\end{equation}

Using (3.4) and (3.5), the equation (3.7) becomes

\begin{equation}
\epsilon((\nabla w Q_h u, \nabla w v)_{\mathcal{T}_h} + (\nabla w \cdot (bQ_h u), v_0)_{\mathcal{T}_h} + (cQ_0 u, v_0) = (f, v_0) + \ell_a(u, v) + \ell_b(u, v) + \ell_c(u, v).
\end{equation}

Adding $s_c(Q_h u, v)$ and $s_d(Q_h u, v)$ to both sides of the above equation gives

\begin{equation}
a(Q_h u, v) = (f, v_0) + \ell_a(u, v) + \ell_b(u, v) + \ell_c(u, v) + s_c(Q_h u, v) + s_d(Q_h u, v).
\end{equation}

Subtracting (2.8) from (3.8) yields the error equation (3.6). This completes the proof of the lemma. □

### 3.2. Error estimates.

We will derive error estimates in this section. For any function $\varphi \in H^1(T)$, the following trace inequality holds true.

\begin{equation}
\|\varphi\|_{\ast}^2 \leq C (h^{-1}_T\|\varphi\|_{T}^2 + h_T\|\nabla \varphi\|_{T}^2).
\end{equation}

**Lemma 3.6.** Let $u$ be the solution of the problem (1.1)-(1.2) and $\mathcal{T}_h$ be a finite element partition of $\Omega$ satisfying the shape regularity assumptions A1 - A4. Then, the $L^2$ projections $Q_0$ and $Q_h$ satisfy

\[
\sum_{T \in \mathcal{T}_h} \left( \|u - Q_0 u\|_{T}^2 + h_T^2 \|\nabla (u - Q_0 u)\|_{T}^2 \right) \leq Ch^{2(s+1)} \|u\|_{s+1}^2, \quad 0 \leq s \leq k,
\]

\[
\sum_{T \in \mathcal{T}_h} \left( \|\nabla u - Q_h \nabla u\|_{T}^2 + h_T^2 \|\nabla (u - Q_h \nabla u)\|_{1, T}^2 \right) \leq Ch^{2s} \|u\|_{s+1}^2, \quad 0 \leq s \leq k.
\]

**Proof.** In order to avoid repetition, we refer the readers to the proof of [39, Lemma 4.1]. □
Lemma 3.7. Let \( u \) be the solution of the problem (1.1)-(1.2). Then for \( v \in V_h \),

\[
(3.10) \quad |\ell_a(u, v)| \leq C C^2 h^k |u|_{k+1} v||
\]

\[
(3.11) \quad |\ell_b(u, v)| \leq C h^{k+\frac{1}{2}} |u|_{k+1} v||
\]

\[
(3.12) \quad |\ell_c(u, v)| \leq C h^{k+1} |u|_{k+1} v||
\]

\[
(3.13) \quad |s_c(Q_h u, v)| \leq C h^{k+\frac{1}{2}} |u|_{k+1} v||
\]

\[
(3.14) \quad |s_d(Q_h u, v)| \leq C C^2 h^k |u|_{k+1} v||
\]

Proof. It follows from the Cauchy-Schwarz inequality, the trace inequality (3.9) and Lemma 3.6 that

\[
|\ell_a(u, v)| \leq \sum_{T \in T_h} |\langle (\nabla u - \nabla u) \cdot n, (v_0 - v_b) \rangle_T|
\]

\[
\leq C \sum_{T \in T_h} \epsilon \|\nabla u - \nabla u\| \|v_0 - v_b\|_{\partial T}
\]

\[
\leq C \left( \sum_{T \in T_h} h_T \|\nabla u - \nabla u\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \right)^{\frac{1}{2}}
\]

\[
\leq C h^k |u|_{k+1} \left( \sum_{T \in T_h} \epsilon h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \right)^{\frac{1}{2}}
\]

\[
\leq C h^k |u|_{k+1} v||
\]

To prove (3.11), we need bound \( \ell_1(u, v) \) and \( \ell_2(u, v) \) by definition of \( \ell_b(u, v) \). Let \( \overline{b} \) be a piecewise constant function whose value is the average of \( b \) over the element. It follows from the Lemma 3.6 and the inverse inequality,

\[
|\ell_1(u, v)| = |\langle (u - Q_0 u, b \cdot \nabla v_0) \rangle_{\partial T_h}| = |\langle (u - Q_0 u, (b - \overline{b}) \cdot \nabla v_0) \rangle_{\partial T_h}|
\]

\[
\leq C h^{k+1} |u|_{k+1} v||
\]

On the other hand, it follows from the definition of \( Q_b \), Lemma 3.6 and Cauchy-Schwarz inequality,

\[
|\ell_2(u, v)| = |\langle (u - Q_b u, b \cdot n(v_0 - v_b)) \rangle_{\partial T_h}|
\]

\[
\leq C h^{k+\frac{1}{2}} |u|_{k+1} v||
\]

where we use the fact \( \|u - Q_0 u\|_{\epsilon} \leq \|u - Q_0 u\|_{\epsilon} \).

Combining the two estimates above, we have proved (3.11). Similarly, we can obtain (3.12).

It follows from (3.9), Cauchy-Schwarz inequality and Lemma 3.6,

\[
|s_c(Q_h u, v)| \leq \sum_{T \in T_h} |\langle b \cdot n(Q_0 u - Q_b u), v_0 - v_b \rangle |_{\partial T}
\]

\[
\leq \left( \sum_{T \in T_h} \|b \cdot n\|_{\partial T} \|Q_0 u - u + u - Q_b u\|_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \|b \cdot n\|_{\partial T} \|v_0 - v_b\|_{\partial T} \right)^{\frac{1}{2}}
\]

\[
\leq C \left( \sum_{T \in T_h} \|b \cdot n\|_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \|b \cdot n\|_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \|b \cdot n\|_{\partial T} \right)^{\frac{1}{2}}
\]

\[
\leq C h^{k+\frac{1}{2}} |u|_{k+1} v||
\]
Similarly, we can obtain (3.14) and complete the proof of the lemma. □

**Theorem 3.8.** Let \( u_h \in V_h^0 \) be the WG finite element solution of the problem (1.1)-(1.2) arising from (2.8). Then there exists a constant \( C \) such that

\[
\| Q_h u - u_h \| \leq C(e^{1/2} + h^{1/2})h^k|u|_{k+1}.
\]

**Proof.** It follows from (3.2) that

\[
\| Q_h u - u_h \|^2 \leq Ca(e_h, e_h).
\]

Letting \( v = e_h \) in (3.6) gives

\[
a(e_h, e_h) = \ell_a(u, e_h) + \ell_b(u, e_h) + \ell_c(u, e_h) + s_c(Q_h u, e_h) + s_d(Q_h u, e_h).
\]

Combining the two equations above with Lemma 3.7, we have

\[
\| Q_h u - u_h \| \leq C(e^{1/2} + h^{1/2})h^k|u|_{k+1}.
\]

We have proved the theorem. □

Next, we discuss the relation between the WG norm \( \| \cdot \| \) defined in (2.9)

\[
\| v \|^2 = \epsilon \sum_{T \in \mathcal{T}_h} \| \nabla w_v \|_T^2 + \| v_0 \|^2 + \sum_{T \in \mathcal{T}_h} \| b \cdot n \|^{1/2}_{\partial T} (v_0 - v_b)\|_{\partial T}^2 + s_d(v, v),
\]

and the SUPG norm \( \| \cdot \|_{SUPG} \) defined as follows for a weak function \( v = \{ v_0, v_b \} \in V_h, \)

\[
\| v \|_{SUPG}^2 = \epsilon \sum_{T \in \mathcal{T}_h} \| \nabla v_0 \|_T^2 + \| v_0 \|^2 + \sum_{T \in \mathcal{T}_h} \delta_T \| b \cdot \nabla v_0 \|_T^2.
\]

The following lemma is helpful to understand the relation between the two norms defined above.

**Lemma 3.9.** There exists a positive constant \( C \) such that for any \( v = \{ v_0, v_b \} \in V_h, \) we have

\[
\sum_{T \in \mathcal{T}_h} \| \nabla v_0 \|_T^2 \leq C \left( \sum_{T \in \mathcal{T}_h} \| \nabla w_v \|_T^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} (v_0 - v_b, v_0 - v_b)_{\partial T} \right).
\]

**Proof.** For any \( v = \{ v_0, v_b \} \in V_h, \) it follows from the definition of weak gradient (2.5) and integration by parts that for any \( q \in \left[ P_{k-1}(T) \right]^d, \)

\[
(\nabla w_v, q)_T = -(v_0, \nabla \cdot q)_T + (v_b, q \cdot n)_{\partial T} = (\nabla v_0, q)_T + (v_b - v_0, q \cdot n)_{\partial T}.
\]

By letting \( q = \nabla v_0 \) in the above identity we arrive at

\[
(\nabla w_v, \nabla v_0)_T = (\nabla v_0, \nabla v_0)_T + (v_b - v_0, \nabla v_0 \cdot n)_{\partial T}.
\]

Thus, from the Cauchy-Schwarz, trace, and inverse inequalities, we have

\[
\| \nabla v_0 \|_T^2 \leq \| \nabla w_v \|_T^2 \| \nabla v_0 \|_T + Ch_T^{-1/2} \| v_0 - v_b \|_{\partial T} \| \nabla v_0 \|_T^2.
\]

This leads to

\[
\| \nabla v_0 \|_T \leq C \left( \| \nabla w_v \|_T^2 + Ch_T^{-1} \| v_0 - v_b \|_{\partial T}^2 \right)^{1/2},
\]
which completes the proof of the lemma. □

Lemma 3.9 implies that for $v = \{v_0, v_b\} \in V_h$,

$$\epsilon \sum_{T \in T_h} \|\nabla v_0\|_T^2 \leq C(\epsilon \sum_{T \in T_h} \|\nabla w v\|_T^2 + s_d(v, v)).$$

**Remark 2.** It follows from (3.18) that the two norms $\| \cdot \|$ and $\| \cdot \|_{SUPG}$ are similar, where the only difference is that the term $\sum_{T \in T_h} \delta_T \|b \cdot \nabla v_0\|_T^2$ in $\| \cdot \|_{SUPG}$ is replaced by $\sum_{T \in T_h} \|b \cdot n\|_1^{1/2}(v_0 - v_b)^2$ in $\| \cdot \|$. This difference reflects different “upwinding” strategies of the SUPG method and the WG method. The SUPG method adds a residual term to the continuous finite element method to provide additional control on the convective derivative in the streamline direction. On the other hand, the term $\sum_{T \in T_h} \|b \cdot n\|_1^{1/2}(v_0 - v_b)^2$ in $\| \cdot \|$ represents the upwinding strategy of using discontinuous approximation.

4. Numerical example. In this section, we test numerically the performance of the proposed WG FEM for solving boundary value problems (1.1)-(1.2). Examples of different types have been tested to confirm the efficiency of the method. Numerical errors will be measured in the $L^2$ norm $\| \cdot \|$ and/or the energy norm $\| \cdot \|$.

4.1. Example 1: Smooth solution. This example is adopted from [2]. Let $\Omega = (0, 1)^2$, $b = (1, 1)^T$, and $c = 0$. The source term $f$ is chosen so that the exact solution is

$$u(x, y) = \sin(2\pi x) \sin(2\pi y).$$

We computed approximate solutions to the boundary value problem for $\epsilon = 10^{-3}$ and $10^{-9}$ to confirm that the WG scheme is valid. Triangular meshes are used. Convergence plots for numerical errors for linear, quadratic, and cubic elements have been collected in Figures 4.1. It is observed that optimal $L^2$ estimate of $O(h^{k+1})$ is obtained. On the other hand, $\|Q_h u - u_h\|$ converges at $O(h^{k+1/2})$, which matches the estimation in Theorem 3.8.

4.2. Example 2: Boundary layer. In this example, we examine the performance of the proposed WG method in the occurrence of boundary layer. Let $\Omega = (0, 1)^2$, $b = (1, 1)^T$, and $c = 0$ in (1.1). We choose $f$ and an appropriate Dirichlet boundary condition so that the exact solution is

$$u(x, y) = \sin \frac{\pi x}{2} + \sin \frac{\pi y}{2} + \frac{e^{-1/\epsilon} - e^{-1/(x-1)(y-1)/\epsilon}}{(1 - e^{-1/\epsilon})}.$$

$P_2$ finite elements are used on rectangular uniform meshes. Numerical errors measured in different norms and their convergence rates are provided in Tables 4.1 and 4.2. In particular, as shown in Table 4.1, the optimal convergence rate of $O(h^{k+1/2})$ in Theorem 3.8 is confirmed. For comparison, we measure also the errors of the WG finite element solutions in the standard SUPG energy norm [19] defined by (3.16) (see also [34]). In Table 4.1, we present the numerical results with $\delta_T = h_T$ in (3.16). It is observed that the convergence order of $Q_h u - u_h$ in the SUPG norm is the same as that measured in the WG energy norm.

We shall report that, as other upwind schemes (see, e.g. [34]), our method has disappointing performance inside the layer; cf. also Remark 1. In Table 4.2, it is observed that the numerical behavior of the method is poor in the intermediate regimes (e.g. when $\epsilon = 10^{-3}$), which however is very good in the strongly advection-dominated
Fig. 4.1. Example 1. Convergence rates of $\|Q_0 u - u_0\|$ and of $|||Q_h u - u_h|||$ of $P_1$, $P_2$, and $P_3$ elements.

### Table 4.1

<table>
<thead>
<tr>
<th>$h^n$</th>
<th>$|Q_h u - u_h|$</th>
<th>$|Q_h u - u_h|_{SUPG}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0692366</td>
<td>0.1214335</td>
</tr>
<tr>
<td>2</td>
<td>0.0177851</td>
<td>0.0269782, 2.26</td>
</tr>
<tr>
<td>3</td>
<td>0.0034200</td>
<td>0.0049298, 2.50</td>
</tr>
<tr>
<td>4</td>
<td>0.0006320</td>
<td>0.0009027, 2.55</td>
</tr>
<tr>
<td>5</td>
<td>0.0001142</td>
<td>0.0001631, 2.52</td>
</tr>
<tr>
<td>6</td>
<td>0.0000204</td>
<td>0.0000292, 2.51</td>
</tr>
</tbody>
</table>

Example 2. The numerical errors and their orders of convergence with $P_2$ elements when $\epsilon = 10^{-9}$.

cases (e.g. when $\epsilon = 10^{-9}$); see also Table 4.1. Similar phenomena in DG method have been reported in the literature (see, e.g., [2, 20]). On the other hand, when $\epsilon = 10^{-9}$, optimal convergence of $Q_0 u - u_0$ in both the $L^2$ norm and the $H^1$ seminorm are obtained.

The plots of $u_h$ for $\epsilon = 10^{-3}$ and $\epsilon = 10^{-9}$ over a mesh of 1,024 uniform elements are shown in Figures 4.2 and 4.3, respectively. In Figures 4.2 and 4.3, $u_0$ is plotted for the numerical solutions $u_h = \{u_0, u_b\}$. For the exact solution $u$, $Q_0 u$ of $Q_h u = \{Q_0 u, Q_b u\}$ is plotted and the average of $Q_0 u$ and $Q_b u$ is plotted on $\partial \Omega$. 
4.3. Example 3: Interior layer – continuous boundary conditions. In this example, we examine the performance of the proposed WG method in the occurrence of interior layer. Let \( \Omega = (0, 1)^2 \), \( b = (1, 0)^T \), and \( c = 1 \). The source term \( f \) is given such that the exact solution is

\[
u(x, y) = 0.5x(1 - x)y(1 - y) \left( 1 - \tanh \frac{\alpha - x}{\gamma} \right).
\]

Here the parameters \( \alpha \) and \( \gamma \) control the location and thickness of the interior layer. In Figures 4.4 and 4.5, numerical results of the convection-diffusion-reaction problem with \( \epsilon = 10^{-9} \), \( \alpha = 0.5 \), and \( \gamma = 0.05 \) are provided. When \( \mathcal{P}_1 \) finite element space is used, the numerical errors \( \|Q_h u - u_h\| \) and \( \|Q_0 u - u_0\| \) converge in \( O(h^{3/2}) \) and \( O(h^2) \), respectively. The interior layer can be accurately captured.
Exact solution

WG Solution

(a) Exact solution

(b) Numerical solution over a $32 \times 32 \times 2$ uniform triangle mesh

Fig. 4.4. Example 3. Exact solution and weak Galerkin solution of $P_1$ element with $\epsilon = 10^{-9}$.

Fig. 4.5. Example 3. Convergence rates of $\|Q_h u - u_h\|$ and $\|Q_0 u - u_0\|$ of $P_1$ element with $\epsilon = 10^{-9}$.

4.4. Example 4: Internal layer – discontinuous boundary conditions.

Let $\Omega = (0,1)^2$, $b = (1,1)^T$, and $c = 0$. We consider the following problem

$$-\epsilon \Delta u + \nabla \cdot (bu) + cu = 0 \quad \text{in } \Omega,$$

$$u = g = \begin{cases} 1 & \text{on } \{0\} \times (1/4,1) \\ 0 & \text{elsewhere on } \partial \Omega. \end{cases}$$

Due to the discontinuities in boundary conditions, the solution of this problem is not in $H^1(\Omega)$. Numerical oscillations occur near the interior layer caused by the joints of the conflicting Dirichlet boundary conditions. This phenomenon has been reported in the literature for DG methods; see, e.g., [2, 20].

We use $P_k$ elements over rectangular meshes. The discrete solutions to problem (4.2)-(4.3) for $k = 2$ and $k = 4$ on the mesh of 1,024 uniform squares are plotted in Figure 4.6, in which oscillations are observed near the interior layer. We believe that a better numerical approximation can be obtained if the Dirichlet boundary conditions are enforced weakly. That is, the finite element method (2.8) is modified as: finding $u_h = (u_0, u_b) \in V_h$ such that

$$a(u_h, v) + \frac{\epsilon}{h} (u_b, v_b)_{\partial \Omega} - \epsilon (\nabla u_h \cdot n, v_b)_{\partial \Omega} = (f, v_0) + \frac{\epsilon}{h} (g, v_b)_{\partial \Omega} \quad \forall v \in V_h.$$
The solutions to (4.4) using $P_2$ and $P_4$ elements are shown in Figure 4.7. Improved results are observed. This phenomenon has been discussed in the literature; see, e.g. [35]. For both (2.8) and (4.4), the solutions do not oscillate out of the internal layer.

![Figure 4.6. Example 4. The solution $u_h$ by (2.8) on a mesh of 1,024 uniform squares.](image)

![Figure 4.7. Example 4. The solution $u_h$ by (4.4) on a mesh of 1,024 uniform squares.](image)

### 4.5. Example 5: Rotational flow.
We solve the following boundary value problem

$$-\epsilon \Delta u + \nabla \cdot (bu) + cu = 0 \quad \text{in } \Omega = (0,1)^2 \setminus \left\{ \frac{1}{2} \right\} \times (0, \frac{1}{2}),$$

$$u = x(1-x)y(1-y)(y - \frac{1}{2})^2 \quad \text{on } \partial \Omega,$$

where $\epsilon = 10^{-9}$, $b = \left( \frac{1}{2} - y, x - \frac{1}{2} \right)$, and $c = 10^{-3}$. $P_k$ elements are used on rectangular meshes. The WG solutions for $k = 2$ and $k = 4$ over a mesh of 256 elements are plotted in Figure 4.8. The numerical results demonstrate that the weak Galerkin discretization is stable.

![Figure 4.8. Example 5. The numerical solution $u_h$ on a mesh of 256 uniform squares.](image)
REFERENCES


