A LOCKING FREE REISSNER-MINDLIN ELEMENT WITH WEAK GALERKIN ROTATIONS

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Abstract. A locking free finite element method is developed for the Reissner-Mindlin equations in their primary form. In this method, the transverse displacement is approximated by continuous piecewise polynomials of degree $k+1$ and the rotation is approximated by weak Galerkin elements of degree $k$ for $k \geq 1$. A uniform convergence in thickness of the plate is established for this finite element approximation. The numerical examples demonstrate locking free of the method.

Key words. weak Galerkin, finite element methods, the Reissner-Mindlin plate

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1. Introduction. The Reissner-Mindlin model is frequently used by engineers for plates and shells of small to moderate thickness. This model is well known for its “locking” phenomenon so that many numerical approximations behave poorly when the thickness parameter $t$ tends to zero. There have been extensive efforts on seeking locking free finite element schemes. Locking free elements have been developed on triangles, rectangles and quadrilaterals [3, 12, 7, 21]. Many of these finite element methods are in the mixed forms for examples [6, 9, 8, 11, 17, 20]. Discontinuous Galerkin finite element methods have been applied to solve the Reissner-Mindlin equations [1, 2, 10, 13].

The goal of this paper is to introduce a locking free finite element method. In this method, the rotation is approximated by weak Galerkin elements of degree $k$ while the transverse displacement is approximated by continuous piecewise polynomials of degree $k+1$. The weak Galerkin methods introduced in [18, 19] are new finite element methods that make use of discontinuous polynomials on polytopal meshes. Weak Galerkin finite element methods are the natural extension of the standard Galerkin finite element methods for functions with discontinuities.

We have proved that the method converges with optimal order uniformly with respect to the thickness of the plate. The numerical results also demonstrate that the method is locking free.

The Reissner-Mindlin equations with clamped boundary are considered with the
Let \( \theta = \lambda t^{-2}(\nabla \omega - \theta) \) and the transverse displacement \( w \),

\begin{align}
\text{PDE1} & \quad \nabla \cdot (C \epsilon(\theta)) - \lambda t^{-2}(\nabla \omega - \theta) = 0, \quad \text{in } \Omega, \\
\text{PDE2} & \quad -\nabla \cdot \lambda t^{-2}(\nabla \omega - \theta) = g, \quad \text{in } \Omega, \\
\text{bc} & \quad \theta = 0, \quad w = 0, \quad \text{on } \partial \Omega,
\end{align}

where the load \( g \) is in \( L^2(\Omega) \), \( C \) is the tensor of bending moduli, \( \lambda \) is the shear correction factor and \( t \) is the plate thickness. The domain \( \Omega \) is a convex polygon. For simplicity, let \( \lambda = 1 \). As usual \( \epsilon(\theta) = \frac{1}{2}(\nabla \theta + \nabla \theta^T) \).

2. Finite Element Scheme. Let \( T_h \) be a quasi uniform triangulation of the domain \( \Omega \). Denote by \( \mathcal{E}_h \) the set of all edges in \( T_h \), and let \( \mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega \) be the set of all interior edges. For every element \( T \in T_h \), we denote by \( h_T \) its diameter and mesh size \( h = \max_{T \in T_h} h_T \) for \( T_h \).

Define

\[
(v, w) = \int_{\Omega} v w \, dx, \quad (v, w)_T = \int_T v w \, dx, \quad (v, w)_{\partial T} = \int_{\partial T} v w \, ds,
\]

\[
\|v\|_T^2 = \int_T v^2 \, dx, \quad \|v\|_T^2 = \int_T v^2 \, ds.
\]

For a given integer \( k \geq 1 \), we define a finite element space for the rotation as

\[
\Theta_h = \{ \eta = \{ \eta_0, \eta_b \} : \eta_b|_T \in [P_k(T)]^2, \eta_b|_e \in [P_k(e)]^2, e \in \partial T, \eta_b = 0 \text{ on } \partial \Omega \}
\]

and one for the transverse displacement

\[
W_h = \{ v \in H^1_0(\Omega) : v|_T \in P_{k+1}(T) \}.
\]

**Definition 2.1.** (Weak Symmetric Gradient) A weak symmetric gradient operator on the finite element space \( \Theta_h \), denoted by \( \epsilon_w \), is defined as the unique polynomial \( \epsilon_w(\eta) \in [P_{k-1}(T)]^{2 \times 2} \) satisfying the following equation

\[
(\epsilon_w(\eta), \tau)_K = -\langle \eta_0, \nabla \cdot \tau \rangle_K + \langle \eta_b, \tau n \rangle_{\partial K}, \quad \forall \tau \in [P_{k-1}(K)]^{2 \times 2},
\]

where \( \tau = \frac{1}{2}(\tau + \tau^T) \).

For each element \( T \in T_h \), denote by \( Q_0 \) the \( L^2 \) projections from \( [L^2(T)]^2 \) to \( [P_k(T)]^2 \). Let \( Q_b \) be the \( L^2 \) projections from \( [L^2(e)]^2 \) to \( [P_k(e)]^2 \). Denote by \( Q_b \) the \( L^2 \) projection onto the local weak symmetric gradient space \( [P_{k-1}(T)]^{2 \times 2} \). Define \( Q_h \Theta = \{ Q_0 \theta, Q_b \theta \} \subset \Theta_h \).

Now we introduce two bilinear forms for \( \phi = \{ \phi_0, \phi_b \}, \eta = \{ \eta_0, \eta_b \} \in \Theta_h \),

\[
s(\phi, \eta) = \sum_{T \in T_h} h^{-1}_T (\phi_0 - \phi_b, \eta_0 - \eta_b)_{\partial T},
\]

\[
a(\phi, \eta) = \sum_{T \in T_h} (\langle \epsilon_w(\phi), \epsilon_w(\eta) \rangle_T + s(\phi, \eta)).
\]

**Algorithm 2.1.** A numerical approximation for (1.1)-(1.3) can be obtained by seeking \( \theta_h = \{ \theta_0, \theta_b \} \in \Theta_h \) and \( w_h \in W_h \) satisfying the following equation for \( \eta = \{ \eta_0, \eta_b \} \in \Theta_h \) and \( v \in W_h \),

\[
a(\theta_h, \eta) + t^{-2}(\nabla w_h - \theta_0, \nabla v - \eta_0) = (g, v).
\]
For any $\eta \in \Theta_h$, define
\begin{equation}
\|\eta\|^2 = a(\eta, \eta).
\end{equation}

For any function $\varphi \in H^1(T)$, the following well known trace inequality holds true,
\begin{equation}
\|\varphi\|_{e}^2 \leq C \left( h_T^{-1} \|\varphi\|_T^2 + h_T \|\nabla \varphi\|_T^2 \right).
\end{equation}

Let elements $T_1$ and $T_2$ have $e$ as a common edge and $n_1$ and $n_2$ be the unit outward normals of $T_1$ and $T_2$ on $e$ respectively. For any $x \in e$, define $\eta_0|_{\partial T_i}(x) = \lim_{\epsilon \downarrow 0} \eta_0(x - \epsilon n_i)$ for $i = 1, 2$. We define the jump and average of $\eta_0$ with $\eta = \{\eta_0, \eta_b\} \in \Theta_h$ on $e$ as
\[
[\eta_0]_e = \left\{ \frac{\eta_0|_{\partial T_1} - \eta_0|_{\partial T_2}}{n_1}, \begin{array}{c} e \in \mathcal{E}_h^0, \\ e \in \partial \Omega \end{array} \right\}, \quad \{\{\eta_0\}\}_e = \left\{ \frac{1}{2}(\eta_0|_{\partial T_1} + \eta_0|_{\partial T_2}), \begin{array}{c} e \in \mathcal{E}_h^0, \\ e \in \partial \Omega \end{array} \right\}.
\]

The order of $T_1$ and $T_2$ is non-essential as long as the difference is taken in a consistent way.

By simple algebraic manipulation, we have the following inequality,
\begin{equation}
s(\eta, \eta) = \sum_{T \in T_h} h^{-1}\|\eta_0 - \eta_b\|_{\partial T}^2 \geq \frac{1}{2} \left( \sum_{e \in \mathcal{E}_h} h^{-1}(\|\{\eta_0\}\|_e^2 + \|[\eta_0]\|_e^2) \right),
\end{equation}

which implies
\begin{equation}
\sum_{e \in \mathcal{E}_h} h^{-1}\|[\eta_0]\|_e^2 \leq Cs(\eta, \eta) \leq C\|\eta\|.
\end{equation}

**Lemma 2.1.** For any $\eta = \{\eta_0, \eta_b\} \in \Theta_h$, we have
\begin{align*}
\sum_{T \in T_h} \|\epsilon(\eta_0)\|_T^2 &\leq C\|\eta\|^2, \\
\sum_{T \in T_h} \|\epsilon_w(\eta)\|_T^2 &\leq C\left( \sum_{T \in T_h} \|\nabla \eta_0\|_T^2 + \sum_{T \in T_h} h_T^{-1}\|\eta_0 - \eta_b\|_{\partial T}^2 \right), \\
\sum_{T \in T_h} \|\nabla \eta_0\|_T^2 &\leq C\|\eta\|^2, \\
\|\eta_0\| &\leq C\|\eta\|.
\end{align*}

**Proof.** Using integration by parts, (2.3), (2.6) and the inverse inequality, we have
\[
(\epsilon(\eta_0), \epsilon(\eta_0))_T = (\epsilon(\eta_0), \nabla \eta_0)_T
= - (\nabla \cdot \epsilon(\eta_0), \eta_0)_T + (\epsilon(\eta_0) \nabla, \eta_0)_{\partial T}
= (\epsilon_w(\eta), \epsilon(\eta_0))_T + (\epsilon(\eta_0) \nabla, \eta_0 - \eta_b)_{\partial T}
\leq C(\|\epsilon_w(\eta)\|_T + h_T^{-1/2}\|\eta_0 - \eta_b\|_{\partial T}^2)\|\epsilon(\eta_0)\|_T,
\]

which implies
\[
\sum_{T \in T_h} \|\epsilon(\eta_0)\|_T^2 \leq C \left( \sum_{T \in T_h} \|\epsilon_w(\eta)\|_T^2 + \sum_{T \in T_h} h_T^{-1}\|\eta_0 - \eta_b\|_{\partial T}^2 \right) \leq C\|\eta\|^2.
\]
Similarly, we can prove
\[
\sum_{T \in \mathcal{T}_h} \| \epsilon_w(\eta) \|^2_T \leq C \left( \sum_{T \in \mathcal{T}_h} \| \epsilon(\eta_0) \|^2_T + \sum_{e \in \mathcal{E}_h} h^{-1} \| \eta_0 - \eta_b \|^2_{\partial T} \right)
\leq C \left( \sum_{T \in \mathcal{T}_h} \| \nabla \eta_0 \|^2_T + \sum_{T \in \mathcal{T}_h} h^{-1} \| \eta_0 - \eta_b \|^2_{\partial T} \right).
\]

For \( \eta = \{ \eta_0, \eta_b \} \in \Theta_h \), the following estimate can be found for example in [1, 5],
\[
\sum_{T \in \mathcal{T}_h} \| \nabla \eta_0 \|^2_T \leq C \left( \sum_{T \in \mathcal{T}_h} \| \epsilon(\eta_0) \|^2_T + \sum_{e \in \mathcal{E}_h} h^{-1} \| \eta_0 \|^2_{\partial T} \right).
\]

Using (2.13), (2.9) and (2.8), we obtain (2.11),
\[
\sum_{T \in \mathcal{T}_h} \| \nabla \eta_0 \|^2_T \leq C \left( \sum_{T \in \mathcal{T}_h} \| \epsilon(\eta_0) \|^2_T + \sum_{e \in \mathcal{E}_h} h^{-1} \| \eta_0 \|^2_{\partial T} \right) \leq C \| \eta \|^2.
\]

To prove (2.12), we need the following estimate which can be found in [15],
\[
\| \eta_0 \|^2 \leq C \left( \sum_{T \in \mathcal{T}_h} \| \nabla \eta_0 \|^2_T + \sum_{T \in \mathcal{T}_h} h^{-1} \| \eta_0 - \eta_b \|^2_{\partial T} \right).
\]

Using the estimate above and (2.11), we proved (2.12) and the lemma. \( \square \)

**Lemma 2.2.** *The finite element formulation (2.4) has a unique solution.*

**Proof.** It suffices to show that zero is the only solution of (2.4) if \( g = 0 \). Letting \( (\eta; v) = (\theta_h; w_h) \) and \( g = 0 \) in (2.4), we have
\[
a(\theta_h, \theta_h) + t^{-2} \sum_{T \in \mathcal{T}_h} \| \nabla w_h - \theta_0 \|_T^2 = 0.
\]

It follows from the fact \( a(\theta_h, \theta_h) = 0 \) and (2.11),
\[
\sum_{T \in \mathcal{T}_h} \| \nabla \theta_0 \|^2_T \leq C \| \theta_h \|^2 = Ca(\theta_h, \theta_h) = 0,
\]
which implies that \( \theta_0 \) is a constant on each \( T \in \mathcal{T}_h \). Combining with \( \theta_0 = \theta_b \) on \( e \in \partial T \) and \( \theta_0 = 0 \) on \( \partial T \), we have \( \theta_h = 0 \). It follows from (2.14) and \( \theta_h = 0 \) that
\[
0 = \| \nabla w_h - \theta_0 \|^2 = \| \nabla w_h \|^2,
\]
which implies \( w_h = 0 \). \( \square \)

**Lemma 2.3.** *Let \( Q_h \) and \( Q_h \) be the \( L^2 \) projection operators as defined. Then, on each element \( T \in \mathcal{T}_h \), we have the following commutative property
\[
\epsilon_w(Q_h \eta) = Q_h \epsilon(\eta), \quad \forall \eta \in [H^1(T)]^2.
\]

**Proof.** Using (2.3), integration by parts and the definitions of \( Q_h \) and \( Q_h \), we have that for any \( \tau \in [P_{k-1}(T)]^2 \)
\[
(\epsilon_w(Q_h \eta), \tau)_T = -(Q_h \eta, \nabla \cdot \tau)_T + (Q_h \eta, \tau \cdot n)_{\partial T}
= -(\eta, \nabla \cdot \tau)_T + (\eta, \tau \cdot n)_{\partial T}
= (\epsilon(\eta), \tau)_T
= (Q_h \epsilon(\eta), \tau)_T.
\]
which implies the desired identity (2.15). \( \square \)
3. Error Analysis. We start this section by obtaining an error equation. Let \( (\theta; w) \) and \( (\theta_h; w_h) \) be the solution of (1.1)-(1.3) and (2.4) respectively. We introduce the shear stress \( \gamma = t^{-2}(\nabla w - \theta) \) and its approximation \( \gamma_h = t^{-2}(\nabla w_h - \theta_h) \). Let \( w_I \) be the interpolation of \( w \) in \( W_h \). For simplicity of analysis, we assume that the coefficient tensor \( C \) in (1.1) is a piecewise constant with respect to the finite element partition \( T_h \).

Define the error functions between the finite element solution and the \( L^2 \) projection of the exact solution as follows,

\[
e_{h} = Q_{h} \theta - \theta_{h} = \{ Q_{h} \theta - \theta_{0}, \ Q_{h} \theta - \theta_{h} \}, \xi_{h} = w_{I} - w_{h}, \ \zeta_{h} = Q_{0} \gamma - \gamma_{h}.
\]

Define

\[
\ell(\theta, \eta) = \sum_{T \in T_h} \langle C \epsilon(\theta)n - CQ_{h} \epsilon(\theta)n, \eta_{0} - \eta_{h} \rangle_{\partial T}.
\]

**Lemma 3.1.** Let \( (e_{h}, \xi_{h}) \in \Theta_{h} \times W_{h} \) be the error of the finite element solution arising from (2.4). Then, for any \( (\eta; v) \in \Theta_{h} \times W_{h} \) we have

\[
a(e_{h}, \eta) + (\zeta_{h}, \nabla v - \eta_{0}) = \ell(\theta, \eta) + s(Q_{h} \theta, \eta).
\]  

**Proof.** Testing (1.1) by \( \eta_{0} \) of \( \eta = \{ \eta_{0}, \eta_{h} \} \in \Theta_{h} \) we arrive at

\[
\sum_{T \in T_h} \langle C \epsilon(\theta), \epsilon(\eta_{0}) \rangle_{T} - \sum_{T \in T_h} \langle C \epsilon(\theta)n, \eta_{0} - \eta_{h} \rangle_{\partial T} - \langle \gamma, \eta_{0} \rangle = 0,
\]

where we have used the fact that \( \sum_{T \in T_h} \langle C \epsilon(\theta)n, \eta_{0} - \eta_{h} \rangle_{\partial T} = 0 \). For any \( \eta \in \Theta_{h} \), it follows from (2.15), the definition of the weak symmetric gradient (2.3), and integration by parts that

\[
(C \epsilon_{w}(Q_{h} \theta), \epsilon_{w}(\eta))_{T} = (CQ_{h} \epsilon(\theta), \epsilon_{w}(\eta))_{T}
\]

\[
= -(\eta_{0}, \nabla \cdot (CQ_{h} \epsilon(\theta)))_{T} + \langle \eta_{0}, CQ_{h} \epsilon(\theta)n \rangle_{\partial T}
\]

\[
= (\nabla \eta_{0}, CQ_{h} \epsilon(\theta))_{T} - \langle \eta_{0} - \eta_{h}, CQ_{h} \epsilon(\theta)n \rangle_{\partial T}
\]

\[
= (C \epsilon(\theta), \epsilon(\eta_{0}))_{T} - \langle CQ_{h} \epsilon(\theta)n, \eta_{0} - \eta_{h} \rangle_{\partial T}.
\]

Combining (3.3) and (3.2), we have

\[
\sum_{T \in T_h} (C \epsilon_{w}(Q_{h} \theta), \epsilon_{w}(\eta))_{T} - \langle \gamma, \eta_{0} \rangle = \ell(\theta, \eta).
\]

Adding \( s(Q_{h} \theta, \eta) \) to both sides of the above equation gives

\[
a(Q_{h} \theta, \eta) - \langle \gamma, \eta_{0} \rangle = \ell(\theta, \eta) + s(Q_{h} \theta, \eta).
\]

Testing (1.2) by using \( v \in W_{h} \), we arrive at

\[
-(\nabla \cdot \gamma, v) = \langle \gamma, \nabla v \rangle = (g, v).
\]

Adding (3.4) and (3.5), we have

\[
a(Q_{h} \theta, \eta) + (Q_{0} \gamma, \nabla v - \eta_{0}) = (g, v) + \ell(\theta, \eta) + s(Q_{h} \theta, \eta).
\]
Subtracting (2.4) from (3.6) yields the error equation (3.1)

\[ a(e_h, \eta) + (\zeta_h, \nabla v - \eta_0) = \ell(\theta, \eta) + s(Q_h \theta, \eta). \]

This completes the proof. □

Define \( P_0 \theta = \nabla w_I - Q_0 \nabla w + Q_0 \theta \). Then we define another projection to \( \Theta_h \) as

\[ P_h \theta = \{ P_0 \theta, Q_0 \theta \} = Q_h \theta + R_h w, \]

where \( R_h w = \{ \nabla w_I - Q_0 \nabla w, 0 \} \in \Theta_h \).

**Theorem 3.2.** Let \((\theta_h; w_h) \in \Theta_h \times W_h\) be the finite element solution of the problem (1.1)-(1.3) arising from (2.4). Then, there exists a constant \( C \) such that

\[ \begin{align*}
\| Q_h \theta - \theta_h \| + t \| Q_0 \gamma - \gamma_h \| &\leq Ch^{k}(\| w \|_{k+2} + \| \theta \|_{k+1}), \\
\| \nabla (w_I - w_h) \| &\leq Ch^{k}(\| w \|_{k+2} + \| \theta \|_{k+1}).
\end{align*} \]

**Proof.** Letting \((\eta; v) = (P_h \theta - \theta_h; \xi_h) = (e_h + R_h w; \xi_h)\), we have

\[ \begin{align*}
\nabla v - \eta_0 &= \nabla \xi_h - (P_0 \theta - \theta_0) \\
&= \nabla w_I - \nabla w_h - (Q_0 \theta - \theta_0 + \nabla w_I - Q_0 \nabla w) \\
&= Q_0 \nabla w - \nabla w_h - (Q_0 \theta - \theta_0) \\
&= Q_0 (\nabla w - \theta) - (\nabla w_h - \theta_0) \\
&= t^2 (Q_0 \gamma - \gamma_h) = t^2 \zeta_h.
\end{align*} \]

It follows from (3.1) and (3.10) that

\[ a(e_h, e_h + R_h w) + t^2 \| \zeta_h \| = \ell(\theta, e_h + R_h w) + s(Q_h \theta, e_h + R_h w), \]

which implies

\[ \| e_h \|^2 + t^2 \| \zeta_h \|^2 = \ell(\theta, e_h) + \ell(\theta, R_h w) + s(Q_h \theta, e_h) \]

\[ + s(Q_h \theta, R_h w) - a(e_h, R_h w). \]

Next, we will estimate the terms on the right hand side.

Using the definition of \( R_h w \), (2.10) and the trace inequality (2.6), we arrive at

\[ \begin{align*}
\| R_h w \|^2 &\leq C \left( \sum_{T \in T_h} | \nabla w_I - Q_0 \nabla w |_{1,T}^2 + \sum_{T \in \Gamma_h} h^{-1} \| \nabla w_I - Q_0 \nabla w \|_{0,T}^2 \right) \\
&\leq C h^{2k} \| w \|_{k+2}^2.
\end{align*} \]

It follows from the Cauchy-Schwarz inequality and the trace inequality (2.6) that

\[ \begin{align*}
| \ell(\theta, e_h) | &= | \sum_{T \in T_h} (\mathcal{C} e(\theta) n - \mathcal{Q} h e(\theta) n, e_0 - e_b)_{\partial T} | \\
&\leq C \sum_{T \in T_h} \| \mathcal{C} (e(\theta) - Q_h e(\theta)) \|_{\partial T} \| e_0 - e_b \|_{\partial T} \\
&\leq C \left( \sum_{T \in T_h} h \| \mathcal{C} (e(\theta) - Q_h e(\theta)) \|_{0,T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h^{-1} \| e_0 - e_b \|_{0,T}^2 \right)^{1/2} \\
&\leq C h^k \| \theta \|_{k+1} \left( \sum_{T \in T_h} h^{-1} \| e_0 - e_b \|_{0,T}^2 \right)^{1/2} \leq C h^k \| \theta \|_{k+1} \| e_h \|. \]
The trace inequality (2.6) implies
\[
|s(Q_h\theta, e_h)| \leq \sum_{T \in \mathcal{T}_h} h^{-1}(Q_0\theta - Q_h\theta, e_0 - e_h)_{\partial T}\\
\leq C\left(\sum_{T \in \mathcal{T}_h} h^{-1}\|Q_0\theta - \theta\|_{\partial T}^2\right)^{1/2}\left(\sum_{T \in \mathcal{T}_h} h^{-1}\|e_0 - e_h\|_{\partial T}^2\right)^{1/2}\\
\leq Ch^k\|\theta\|_{k+1}\|e_h\|.
\]

Similarly, we have
\[
|s(\theta, R_h w)| \leq Ch^k\|\theta\|_{k+1}\|R_h w\| \leq Ch^{2k}(\|\theta\|_{k+1}^2 \|w\|_{k+2}^2).
\]

and
\[
|s(Q_h\theta, R_h w)| \leq Ch^k\|\theta\|_{k+1}\|R_h w\| \leq Ch^{2k}(\|\theta\|_{k+1}^2 \|w\|_{k+2}^2).
\]

It follows from (3.12) that
\[
|a(e_h, R_h w)| \leq C\|e_h\|\|R_h w\| \leq Ch^k\|w\|_{k+2}\|e_h\|.
\]

Combining all the bounds above with (3.11), we obtain (3.8).

Using the definition of \(\zeta_h\), (3.8) and (2.12), we have
\[
\|Q_0\nabla w - \nabla w_h\|^2 \leq C(t^2\|\zeta_h\|^2 + \sum_{T \in \mathcal{T}_h} \|\theta - \theta_0\|^2_T)\\
\leq C(t^2\|\zeta_h\|^2 + \sum_{T \in \mathcal{T}_h} \|\theta - Q_0\theta\|^2_T + \sum_{T \in \mathcal{T}_h} \|e_0\|^2_T)\\
\leq Ch^{2k}(\|w\|_{k+2}^2 + \|\theta\|_{k+1}^2).
\]

It follows from the estimate above,
\[
\|
abla \zeta_h\| \leq \|Q_0\nabla w - \nabla w_h\| + \|
abla w - Q_0\nabla w\|\\
\leq \|Q_0\nabla w - \nabla w_h\| + \|\nabla w - \nabla w\| + \|\nabla w - Q_0\nabla w\|\\
\leq Ch^k(\|w\|_{k+2} + \|\theta\|_{k+1}).
\]

We complete the proof. \(\square\)

Remark 3.3. The high order WG approximations \((k \geq 3)\) to the Reissner-Mindlin plate problem may not be bounded independently of the plate thickness \(t\). This is because \([3]\)
\[
\|\theta\|_s \leq Ct^{\min\{0.5/2-s\}}, \quad s \in \mathbb{R}.
\]

4. Numerical Experiments. The goal of this section is to numerically validate the theoretical conclusions for Algorithm 2.1 through several computational examples.

The following numerical experiments are performed on the finite element space \(\Theta_h\) and \(W_h\) with \(k = 1\). For all \(2 \times 2\) symmetric matrices \(\tau\), let
\[
\mathbb{C}\tau = \frac{E}{12(1-\nu^2)}[(1-\nu)\tau + \nu tr(\tau)I]
\]
with
\[
\lambda = \frac{Ek}{2(1+\nu)}, \quad \text{and} \quad k = \frac{5}{6}.
\]

We shall measure the following norms defined as follows,

- A discrete $H^1$-norm: $\|\theta\|_\theta := \left( a(\theta, \theta) \right)^{1/2}$,
- Element-based $L^2$-norm: $\|\theta_0\| := \left( \sum_{T \in \mathcal{T}_h} \int_T |\theta_0|^2 \, dx \right)^{1/2}$,
- A discrete $H^1$-norm: $\|\nabla w\|_w := \left( \sum_{T \in \mathcal{T}_h} \|\nabla w\|_T^2 \right)^{1/2}$,
- Element-based $L^2$-norm: $\|w\| := \left( \sum_{T \in \mathcal{T}_h} \int_T |w|^2 \, dx \right)^{1/2}$,

where $\theta = \{\theta_0, \theta_b\} \in \Theta_h$ and $w \in W_h$.

### 4.1. Example 1.
Consider the Reissner-Mindlin problem in the square domain $\Omega = (0,1)^2$. In this test, we test a material with Poisson’s ratio $\nu = 0.3$ and Young’s modulus $E = 1.092 \cdot 10^3 \, N/m^2$. Set the exact solution as follows,

\[
\theta(x,y) = \begin{pmatrix}
y^3(y-1)^3x^2(x-1)^2(2x-1) \\
x^3(x-1)^3y^2(y-1)^2(2y-1)
\end{pmatrix}
\]

and
\[
w(x,y) = \frac{1}{3} x^3(x-1)^3 y^3(y-1)^3 \\
- \frac{2\nu}{5(1-\nu)} [y^3(y-1)^3 x(x-1)(5x^2 - 5x + 1) \\
+ x^3(x-1)^3 y(y-1)(5y^2 - 5y + 1)].
\]

The body load is given by
\[
g(x,y) = \frac{E}{12(1-\nu^2)} [12y(y-1)(5x^2 - 5x + 1)(2y^2(y-1)^2 + x(x-1)(5y^2 - 5y + 1)) \\
+ 12x(x-1)(5y^2 - 5y + 1)(2x^2(x-1)^2 + y(y-1)(5x^2 - 5x + 1))].
\]

The error profiles and convergence rate are shown in Table 4.1. It can be obtained that for the linear weak Galerkin element ($k = 1$), the error in $\theta$ and $w$ measured in discrete $H^1$-norm converge at least at the first order and second order, respectively. The superconvergence in the error of $w$ measured in $\|\cdot\|_w$ is observed for this test. Moreover, the errors of $\theta$ and $w$ measured in $L^2$-norm converge in the second order.

### 4.2. Numerical Experiment 2.
Consider the Reissner-Mindlin problem in the unit square $(0,1)^2$ with the loaded force $g = 1$. For this test, we select a material with Poisson’s ratio $\nu = 0.3$ and Young’s modulus $E = 1N/m^2$. Set the exact solution as
Table 4.1  
Example 1. Error profile and convergence rate on triangular mesh.

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<tr>
<th>$h$</th>
<th>$|\theta|_{|Q|_h}$ Rate</th>
<th>$|\theta|_{|Q|_h}$ Rate</th>
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$t = 1e - 3$

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follows

$\theta(x, y) = \left( \frac{x(x^2 + y^2 - 1)}{(x^2 + y^2 - 1)} \right) / 16D$

and

$w(x, y) = \left( \frac{(x^2 + y^2)^2}{64D} - (x^2 + y^2) \left( \frac{t^2}{4\lambda} + \frac{1}{32D} \right) \right) + \frac{t^2}{4\lambda} + \frac{1}{64D},$

where $D = E/[12(1 - \nu^2)].$

Table 4.2 reports the error profiles and convergence rate test. It illustrates that for the linear ($k = 1$) weak Galerkin element, the error in $\theta$ and $w$ measured in discrete $H^1$-norm converge at least at the first order and second order. Moreover, the errors of $\theta$ and $w$ measured in $L^2$-norm converge in the second order. These results confirm the previous theoretical conclusions.

4.3. Numerical Experiment 3. Let $\Omega = (0, 1) \times (0, 1)$, and load function $g(x, y)$ is chosen to satisfy equation (1.1)-(1.2) such that the exact solutions are as follows,

$\theta(x, y) = \left( \frac{2(x - 1)x(2x - 1)(y - 1)^2y^2}{(2y - 1)y(2y - 1)(x - 1)x^2} \right),$

and

$w = \left( \frac{t^2}{5(\nu - 1)} \right) (12x(x - 1)y(y - 1)(x^2 - x + y^2 - y) + 2(x^2(x - 1)^2 + y^2(y - 1)^2)) + x^2(x - 1)^2y^2(y - 1)^2.$
Table 4.2
Example 2. Error profile and convergence rate on triangular mesh.

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<th>$|w|_{H^2}$ Rate</th>
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Furthermore, we test the problem with parameters $\nu = 1/3$ and $E = 1N/m^2$.

Table 4.3 reports error profiles and convergence rate for $\theta$ and $w$. Again, as the results in Section 4.1-4.2, similar conclusions of the convergence results measured in $H^1$-norm and $L^2$-norm can be made for this numerical experiment.

REFERENCES

Table 4.3
Example 3. Error profile and convergence rate on triangular mesh.

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<tr>
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<th>$|e_0|_{Q_0}$</th>
<th>Rate</th>
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$\|Q_0\|_{\theta}$

$t = 1$

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$\|Q_0\|_{\theta}$

$t = 1 \cdot 10^{-3}$

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