A DISCONTINUOUS LEAST-SQUARES FINITE ELEMENT
METHOD FOR SECOND ORDER ELLIPTIC EQUATIONS

XIU YE* AND SHANGYOU ZHANG†

Abstract. In this paper, a discontinuous least-squares (DLS) finite element method is intro-
duced. The novelty of this work is twofold, to develop a DLS formulation that works for general
polytopal meshes and to provide rigorous error analysis for it. This new method provides accurate
approximations for both the primal and the flux variables. We obtain optimal order error estimates
for both the primal and the flux variables. Numerical examples are tested for polynomials up to
degree 4 on non-triangular meshes, i.e., on rectangular and hexagonal meshes.

Key words. discontinuous Galerkin, finite element methods, least-squares finite element meth-
ods, second-order elliptic problems, polygonal mesh

AMS subject classifications. Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50

1. Introduction. We consider the model problem: seeking an unknown function

\[ \begin{align*}
-\nabla \cdot (a \nabla u) + cu &= f, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*} \]

where \( c \geq 0 \) and \( \Omega \) is a polytopal domain in \( \mathbb{R}^d \) with \( d = 2, 3 \), \( \nabla u \) denotes the gradient of the function \( u \), and \( a \) is a \( d \times d \) tensor that is uniformly symmetric positive-definite in the domain. The partial differential equation (1.1) is a benchmark testing problem
for new discretization techniques.

The goal of this work is to develop a discontinuous least-squares finite element
method for the second order elliptic problem (1.1)-(1.2) and provide rigorous error
analysis for the method. This new DLS finite element method has two unique features:
1. provide accurate approximation for both the primal and the flux variables and lead
to a positive and definite system, 2. allow the use of general polytopal meshes.

The least-squares finite element method is a discretization technique for solv-
ing partial differential equations. The method receives its name by minimizing the
residuals in a least-squares fashion. Least-squares finite element methods have been
developed for the second-order elliptic problems in [3, 6, 8, 9, 13, 17] and references
therein. The research of finite element methods with discontinuous approximations
has received extensive attention in the past two decades. Thousands of papers have
been published on discontinuous Galerkin techniques; a few representatives include in-
terior penalty discontinuous Galerkin method [1], local discontinuous Galerkin method

Discontinuous least-squares methods [14, 15] have been developed for singularly
perturbed reaction-diffusion problems. These methods are the first order methods

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with nonsymmetric system. Discontinuous least-squares methods have been developed in [4, 5] for the Stokes equations. Optimal or near optimal convergence rates of the methods are only confirmed numerically. In [2], a discontinuous least-squares method is introduced for the div-curl system on tetrahedral mesh. Most of existing least squares finite element methods can only be applied on simplicial mesh such as triangular or quadrilateral mesh.

Recently, a weak Galerkin least squares finite element method has been developed in [16] for the second order elliptic problems, which can work on general polytopal mesh. Weak Galerkin methods refer to general finite element techniques for partial differential equations involving novel concepts of weak function and weak derivative introduced first in [19, 18] for second order elliptic equations. Motivated by the work in [16], we extend the results of the weak Galerkin method to the discontinuous Galerkin method in this paper. We develop and analyze a discontinuous Galerkin least squares method on polygonal/polyhedral mesh. To the best of our knowledge, this new method is the first discontinuous Galerkin high order polygonal least-squares finite element method for the second order elliptic equations with rigorous error estimates.

2. Discontinuous Galerkin Least Squares Method. Let \( T_h \) be a partition of the domain \( \Omega \) into polygons in 2D or polyhedra in 3D. Assume that \( T_h \) is shape regular in the sense as defined in [18]. Denote by \( \mathcal{E}_h \) the set of all edges or flat faces in \( T_h \), and let \( \mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega \) be the set of all interior edges or flat faces. Let \( \Gamma_h \) be the subset of \( \mathcal{E}_h \) of all edges or faces on \( \Gamma = \partial \Omega \). For every element \( T \in T_h \), we denote by \( h_T \) its diameter and mesh size \( h = \max_{T \in T_h} h_T \) for \( T_h \).

A mixed form of the problem (1.1)-(1.2) can be stated as: Find \( q = q(x) \) and \( u = u(x) \) satisfying

\[
\begin{align*}
q + a \nabla u &= 0, \quad \text{in } \Omega, \\
\nabla \cdot q + cu &= f, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

We now introduce two finite element spaces: \( V_h \) for the pressure variable \( u \) and \( \Sigma_h \) for the flux variable \( q \) defined as follows

\[
\begin{align*}
V_h &= \{ v \in L^2(\Omega) : v|_T \in P_k(T), \forall T \in T_h \}, \\
\Sigma_h &= \{ \sigma \in [L^2(\Omega)]^d : \sigma|_T \in [P_k(T)]^d, \forall T \in T_h \},
\end{align*}
\]

where \( k \geq 1 \) is any nonnegative integer.

Let elements \( T_1 \) and \( T_2 \) have \( e \) as a common edge with \( n_1 \) and \( n_2 \) as the unit outward normal respectively. We define the jump and the average for a scalar function \( v \) on \( e \) as

\[
\begin{align*}
[v]_e &= \begin{cases} 
  v|_{\partial T_1} n_1 + v|_{\partial T_2} n_2, & e \in \mathcal{E}_h^0, \\
v n, & e \in \Gamma_h, 
\end{cases} \\
\{ v \}_e &= \begin{cases} 
  \frac{1}{2} v|_{\partial T_1} + v|_{\partial T_2}, & e \in \mathcal{E}_h^0, \\
v, & e \in \Gamma_h.
\end{cases}
\end{align*}
\]
We define the jump and the average for a vector function $\sigma$ on $e$ as

$$
[e] = \begin{cases} 
\sigma_{|\partial T_1} \cdot n_1 + \sigma_{|\partial T_2} \cdot n_2, & e \in E_h^0, \\
\sigma \cdot n_e, & e \in \Gamma_h,
\end{cases}
$$

$$
\{\sigma\} = \begin{cases} 
\frac{1}{2} (\sigma_{|\partial T_1} + \sigma_{|\partial T_2}), & e \in E_h^0, \\
\sigma, & e \in \Gamma_h.
\end{cases}
$$

First, we introduce some notations,

$$
(v, w)_T = \sum_{T \in \mathcal{T}_h} (v, w)_T = \sum_{T \in \mathcal{T}_h} \int_T vwdx,
$$

$$
\langle v, w \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \langle v, w \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} vwds,
$$

$$
\langle v, w \rangle_{E_h} = \sum_{e \in E_h} \langle v, w \rangle_e = \sum_{e \in E_h} \int_e vwds.
$$

Then we define a bilinear form,

$$
A(q_h, u_h; \sigma, v) = \langle \nabla \cdot \tau + cw, \nabla \cdot \sigma + cv \rangle_{\mathcal{T}_h} + \langle \tau + a \nabla w, \sigma + a \nabla v \rangle_{\mathcal{T}_h} + s_1(w, v) + s_2(\tau, \sigma),
$$

where

$$
s_1(w, v) = \langle h_e^{-1} [w], [v] \rangle_{E_h}, \quad s_2(\tau, \sigma) = \langle h_e^{-1} [\tau], [\sigma] \rangle_{E_h}.
$$

**Algorithm 1.** The discontinuous Galerkin least-squares method for the problem (2.1)-(2.3) seeks $u_h \in V_h$ and $q_h \in \Sigma_h$ satisfying

$$
A(q_h, u_h; \sigma, v) = (f, \nabla \cdot \sigma + cv), \quad \forall \sigma \times v \in \Sigma_h \times V_h.
$$

**3. Existence and Uniqueness.** Let $T$ be an element with $e$ as an edge. For any function $\varphi \in H^1(T)$, the following trace inequality holds true,

$$
\|\varphi\|_{\partial T}^2 \leq C (h_e^{-1} \|\varphi\|_T^2 + h_T \|\nabla \varphi\|_T^2).
$$

Define

$$
|v|_{1,h}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla v\|_T^2.
$$

Introduce a norm $\|\cdot\|_V$ for $V_h$ and a norm $\|\cdot\|_{\Sigma}$ for $\Sigma_h$ as follows

$$
\|v\|_V^2 = |v|_{1,h}^2 + s_1(v, v),
$$

$$
\|\sigma\|_{\Sigma}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla \cdot \sigma\|_T^2 + \|\sigma\|_T^2 + s_2(\sigma, \sigma).
$$

The following discrete Poincaré inequality has been established in [7],

$$
\|v\| \leq C \|v\|_V, \quad v \in V_h.
$$
Lemma 3.1. There exists a constant $C$ such that for all $\sigma \times v \in \Sigma_h \times V_h$ one has

\begin{equation}
A(\sigma, v; \sigma, v) \geq C(\|\sigma\|^2_\Sigma + \|v\|^2_V).
\end{equation}

Proof. First, note that

\begin{align}
(3.4) \quad s_1(v, v) & \leq A(\sigma, v; \sigma, v), \\
(3.5) \quad s_2(\sigma, \sigma) & \leq A(\sigma, v; \sigma, v).
\end{align}

It follows from the integration by parts,

\begin{equation}
-(\nabla v, \sigma) = (v, \nabla \cdot \sigma)_{\partial T_h} - \langle v, \sigma \cdot n \rangle_{\partial T_h}
\end{equation}

\begin{equation}
= (\nabla \cdot \sigma + cv, v)_{\partial T_h} - (cv, v)_{\partial T_h} - \langle v, \sigma \cdot n \rangle_{\partial T_h}.
\end{equation}

Using the trace and the inverse inequalities, we have that for $\sigma \times v \in \Sigma_h \times V_h$,

\begin{align}
(3.7) \quad \langle v, \sigma \cdot n \rangle_{\partial T_h} &= \langle \{v\}, \{\sigma\}\rangle_{\Sigma} + \langle \{v\}, [\sigma]\rangle_{\Sigma_h} \xi_h \\
& \leq C(s_1^{1/2}(v, v)\|\sigma\| + s_2^{1/2}(\sigma, \sigma)\|v\|).
\end{align}

The equation (3.6) implies

\begin{align}
\sum_{T \in T_h} \|a^{1/2} \nabla v\|^2_T &= (a \nabla v + \sigma, \nabla v)_{\partial T_h} - \langle \sigma, \nabla v \rangle_{\partial T_h} \\
&= (a \nabla v + \sigma, \nabla v)_{\partial T_h} + (\nabla \cdot \sigma + cv, v)_{\partial T_h} - (cv, v)_{\partial T_h} \\
&\quad - \langle v, \sigma \cdot n \rangle_{\partial T_h} \\
&\leq (a \nabla v + \sigma, \nabla v)_{\partial T_h} + (\nabla \cdot \sigma + cv, v)_{\partial T_h} - \langle v, \sigma \cdot n \rangle_{\partial T_h},
\end{align}

and

\begin{align}
\|a^{-1/2} \sigma\|^2 &= (\sigma + a \nabla v, a^{-1} \sigma)_{\partial T_h} - (\nabla v, \sigma)_{\partial T_h} \\
&= (\sigma + a \nabla v, a^{-1} \sigma)_{\partial T_h} + (\nabla \cdot \sigma + cv, v)_{\partial T_h} - (cv, v)_{\partial T_h} \\
&\quad - \langle v, \sigma \cdot n \rangle_{\partial T_h} \\
&\leq (\sigma + a \nabla v, a^{-1} \sigma)_{\partial T_h} + (\nabla \cdot \sigma + cv, v)_{\partial T_h} - \langle v, \sigma \cdot n \rangle_{\partial T_h}.
\end{align}

Using (3.2) and (3.7), we have

\begin{align}
\|v\|^2_V + \|\sigma\|^2 & \leq C \left( \sum_{T \in T_h} \|a^{1/2} \nabla v\|^2_T + \|a^{-1/2} \sigma\|^2 \right) + s_1(v, v) \\
& \leq C((\sigma + a \nabla v, \nabla v)_{\partial T_h} + (\sigma + a \nabla v, a^{-1} \sigma)_{\partial T_h} \\
&\quad + 2(\nabla \cdot \sigma + cv, v)_{\partial T_h} - 2(v, \sigma \cdot n)_{\partial T_h}) + s_1(v, v) \\
& \leq CA^{1/2}(\sigma, v; \sigma, v)(\|v\|^2_V + \|\sigma\|^2) + A(\sigma, v; \sigma, v) \\
& \leq CA(\sigma, v; \sigma, v) + \frac{1}{4}(\|v\|^2_V + \|\sigma\|^2) + A(\sigma, v; \sigma, v) \\
& \leq CA(\sigma, v; \sigma, v) + \frac{1}{2}(\|v\|^2_V + \|\sigma\|^2).\n\end{align}
which implies

\[(3.8) \quad \|\varepsilon\|_V^2 + \|\sigma\|^2 \leq CA(\sigma, v; \sigma, v).\]

Similarly, we have from the estimates (3.2) and (3.8) that

\[\|\nabla \cdot \sigma\|^2 \leq C(\|\nabla \cdot \sigma + cv\|^2 + \|v\|^2) \leq CA(\sigma, v; \sigma, v).\]

Combining the two estimates above with (3.5) gives

\[\|\sigma\|^2 \leq CA(\sigma, v; \sigma, v),\]

which, together with (3.8), completes the proof of the lemma. \[\square\]

**Lemma 3.2.** The discontinuous least-squares finite element scheme (2.6) has one and only one solution.

**Proof.** It suffices to prove the uniqueness. If \(q_h^{(1)} \times u_h^{(1)}\) and \(q_h^{(2)} \times u_h^{(2)}\) are two solutions of (2.6), then \(\tau_h = u_h^{(1)} - u_h^{(2)}\) and \(\eta_h = q_h^{(1)} - q_h^{(2)}\) would satisfy the following equation

\[A(\eta_h, \tau_h; \sigma, v) = 0, \quad \forall \sigma \times v \in \Sigma_h \times V_h.\]

Note that \(\tau_h \in V_h\). By letting \(v = \tau_h\) and \(\sigma = \eta_h\) in the above equation we have

\[\|\tau_h\|^2 + \|\eta_h\|^2 \leq CA_h(\eta_h, \tau_h; \eta_h, \tau_h) = 0,\]

which implies \(\tau_h \equiv 0\) and \(\eta_h \equiv 0\) or equivalently, \(u_h^{(1)} \equiv u_h^{(2)}\) and \(q_h^{(1)} \equiv q_h^{(2)}\). This completes the proof of the lemma. \[\square\]

**4. Error Analysis.** Let \(q_h \times u_h \in \Sigma_h \times V_h\) be the discontinuous least-squares finite element solution arising from (2.6), and \(Q_h q \times Q_h u \in \Sigma_h \times V_h\) be the \(L^2\) projection of the exact solution \(q \times u\) defined element-wise. Their differences are referred to as the error functions, and they are denoted as

\[(4.1) \quad \varepsilon_h = Q_h q - q_h, \quad e_h = Q_h u - u_h.\]

**Lemma 4.1.** Assume that \(T_h\) is shape regular. Then for \(u \in H^{k+1}(\Omega)\) and \(q \in [H^{k+1}(\Omega)]^d\), we have

\[(4.2) \quad |s_1(Q_h u, v)| \leq Ch^k \|u\|_{k+1} \|v\|_V,\]

\[(4.3) \quad |s_2(Q_h q, \sigma)| \leq Ch^k \|q\|_{k+1} \|\sigma\|_{\Sigma}.\]

**Proof.** First, recall that \(Q_h\) and \(Q_h\) are the \(L^2\) projections defined element-wise onto \(P_h(T)\) and \([P_h(T)]^d\) respectively for each element \(T \in T_h\). We will use approximation properties of standard \(L^2\) projections in the following estimates. Using the trace inequality (3.1), we obtain

\[|s_1(Q_h u, v)| = \langle h^{-1}[Q_h u], [v]\rangle_{E_h} = \langle h^{-1}[Q_h u - u], [v]\rangle_{E_h}\]

\[\leq C \left( \sum_{T \in T_h} (h^{-2}\|Q_h u - u\|_T^2 + \|\nabla (Q_h u - u)\|_T^2) \right)^{\frac{3}{2}} \|v\|_V\]

\[\leq Ch^k \|u\|_{k+1} \|v\|_V.\]
Similarly, we have
\[
|s_2(Q_h, q, \sigma)| = \langle h_e^{-1}(Q_h q - q), [\sigma] \rangle_{E_h}^0
\leq Ch^k \|q\|_k + 1 \|\sigma\|_\Sigma.
\]

**Theorem 4.2.** Let \( q_h \times u_h \in \Sigma_h \times V_h \) be the discontinuous least-squares finite element solution of the problem (2.1)-(2.3) arising from (2.6). Assume the exact solution \( u \in H^{k+1}(\Omega) \) and \( q \in [H^{k+1}(\Omega)]^d \), then
\[
\|u_h - Q_h u\|_V + \|q_h - Q_h q\|_\Sigma \leq Ch^k(\|u\|_{k+1} + \|q\|_{k+1}).
\]

**Proof.** It is obvious that the solution \((u, q)\) satisfies
\[
A(q, u; \sigma, v) = (f, \nabla \cdot \sigma + cv), \quad \forall \sigma \times v \in \Sigma_h \times V_h.
\]
The difference of the above equation and the equation (2.6) gives
\[
A(q - q_h, u - u_h; \sigma, v) = 0, \quad \forall \sigma \times v \in \Sigma_h \times V_h.
\]
It follows from the equation above that
\[
A(e_h, e_h; \sigma, v) = -A(q - Q_h q, u - Q_h u; \sigma, v) \quad \forall \sigma \times v \in \Sigma_h \times V_h.
\]
Letting \( v = e_h \) and \( \sigma = e_h \) in the equation above, we have
\[
A(e_h, e_h; e_h, e_h) = -A(q - Q_h q, u - Q_h u; e_h, e_h).
\]
It then follows from (3.3), (4.2), (4.3) and the definitions of \( Q_h \) and \( Q_h \) that
\[
\|e_h\|_V^2 + \|e_h\|_\Sigma^2 \leq CA(e_h, e_h; e_h, e_h) = CA(q - Q_h q, u - Q_h u; e_h, e_h)
\leq C(|(\nabla \cdot (q - Q_h q) + c(u - Q_h u), \nabla \cdot e_h + ce_h)_{T_h}|
+ |q - Q_h q, a\nabla (u - Q_h u), e_h + a\nabla e_h)_{T_h}|
+ |s_1(Q_h u, e_h)| + |s_2(Q_h q, e_h)|)
\leq Ch^k(\|u\|_{k+1} + \|q\|_{k+1})(\|e_h\|_V + \|e_h\|_\Sigma),
\]
which implies (4.4). This completes the proof. \( \qed \)

**Remark 4.3.** In this paper, both pressure and velocity variables are approximated by \( k \)th order polynomials for \( k \geq 1 \). For this case, it is difficult to prove an optimal \( L^2 \) convergence rate of the pressure variable for the least-squares finite element method. We cannot find similar results in literature. In [17], an optimal \( L^2 \) convergence rate for the pressure is proved when one degree higher polynomials are used for the flux variable. However, the numerical results in the next section demonstrate our method produces an approximation for pressure \( u_h \) with an optimal \( L^2 \) convergence rate.
5. Numerical Test. We numerically solve the boundary value problem (1.1)-(1.2) on the unit square $\Omega = (0, 1) \times (0, 1)$ with the exact solution

\begin{equation}
    u = 2^4 x(1-x)y(1-y),
\end{equation}

where $a = 1$ and $c = 1$. So the exact gradient solution is

\begin{equation}
    q = \left( \begin{array}{c} -2^4(1-2x)y(1-y) \\ 2^4x(1-x)(1-2y) \end{array} \right).
\end{equation}

![Fig. 5.1. The first four grids for numerical solutions in Tables 5.1-5.4.](image)

<table>
<thead>
<tr>
<th>Table 5.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>The errors, $e_u = Q_h u - u_h$ and $e_q = Q_h q - q_h$, and the order of convergence, by the $P_1$ DLS method (2.4) on square grids (Figure 5.1), for (5.1).</td>
</tr>
<tr>
<td>$|e_u|_0$</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
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<tr>
<td>4</td>
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<tr>
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<tr>
<td>6</td>
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<td>7</td>
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<table>
<thead>
<tr>
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</tr>
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<tbody>
<tr>
<td>The errors, $e_u = Q_h u - u_h$ and $e_q = Q_h q - q_h$, and the order of convergence, by the $P_2$ DLS method (2.4) on square grids (Figure 5.1), for (5.1).</td>
</tr>
<tr>
<td>$|e_u|_0$</td>
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</tr>
<tr>
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<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
</tbody>
</table>

In the first set of tests, we use the DLS method (2.4) with polynomial degree $k = 1, 2, 3, 4$ on the uniform grids shown in Figure 5.1. We note that the polynomial space of separated degree $k$, $Q_k$, is used in typical finite element methods, but here we
Table 5.3
The errors, \( e_u = Q_h u - u_h \) and \( e_q = Q_h q - q_h \), and the order of convergence, by the \( P_3 \) DLS method (2.4) on square grids (Figure 5.1), for (5.1).

<table>
<thead>
<tr>
<th></th>
<th>( |e_u|_0 )</th>
<th>( h^n )</th>
<th>( |e_u| )</th>
<th>( h^n )</th>
<th>( |e_q|_0 )</th>
<th>( h^n )</th>
<th>( |e_q| )</th>
<th>( h^n )</th>
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<td>0.976048</td>
<td>1.4577</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.8147</td>
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<td>0.068320</td>
<td>3.8</td>
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<td>3.5</td>
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Table 5.4
The errors and the order of convergence, by the \( P_4 \) DLS method (2.4) on square grids (Figure 5.1), for (5.1).

<table>
<thead>
<tr>
<th></th>
<th>( |Q_h u - u_h|_0 )</th>
<th>( h^n )</th>
<th>( |Q_h u - u_h|_1 )</th>
<th>( h^n )</th>
<th>( |Q_h q - q_h|_0 )</th>
<th>( h^n )</th>
</tr>
</thead>
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<td>0.0</td>
<td>0.0000000000</td>
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</tbody>
</table>

used \( P_k \) polynomials, taking an advantage of the discontinuous Galerkin method. The error and the order of convergence are displayed in Tables 5.1-5.4, where \( Q_h u \) and \( Q_h q \) are the \( L^2 \) projections, and \( u_h \) and \( q_h \) the numerical solutions. The optimal order of convergence is achieved in all cases. In particular, when \( k = 4 \), the exact solution is obtained, up to the computer accuracy. To see conformity, i.e., discontinuity, we plot the finite element solutions on the level 4 square grid in Figure 5.2.

Table 5.5
The errors, \( e_u = Q_h u - u_h \) and \( e_q = Q_h q - q_h \), and the order of convergence, by the \( P_1 \) DLS method (2.4) on hexagon grids (Figure 5.3), for (5.1).

<table>
<thead>
<tr>
<th></th>
<th>( |e_u|_0 )</th>
<th>( h^n )</th>
<th>( |e_u| )</th>
<th>( h^n )</th>
<th>( |e_q|_0 )</th>
<th>( h^n )</th>
<th>( |e_q| )</th>
<th>( h^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.30867</td>
<td>1.9875</td>
<td>2.30605</td>
<td>4.4436</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.17965</td>
<td>0.8</td>
<td>1.5004</td>
<td>0.4</td>
<td>1.09749</td>
<td>1.1</td>
<td>4.1366</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>0.06923</td>
<td>1.4</td>
<td>1.0490</td>
<td>0.5</td>
<td>0.65407</td>
<td>0.7</td>
<td>2.6785</td>
<td>0.6</td>
</tr>
<tr>
<td>4</td>
<td>0.02153</td>
<td>1.7</td>
<td>0.5866</td>
<td>0.8</td>
<td>0.31632</td>
<td>1.0</td>
<td>1.5249</td>
<td>0.8</td>
</tr>
<tr>
<td>5</td>
<td>0.00615</td>
<td>1.8</td>
<td>0.3009</td>
<td>1.0</td>
<td>0.13172</td>
<td>1.3</td>
<td>0.8107</td>
<td>0.9</td>
</tr>
<tr>
<td>6</td>
<td>0.00167</td>
<td>1.9</td>
<td>0.1512</td>
<td>1.0</td>
<td>0.05180</td>
<td>1.3</td>
<td>0.4174</td>
<td>1.0</td>
</tr>
<tr>
<td>7</td>
<td>0.00044</td>
<td>1.9</td>
<td>0.0756</td>
<td>1.0</td>
<td>0.02009</td>
<td>1.4</td>
<td>0.2117</td>
<td>1.0</td>
</tr>
<tr>
<td>8</td>
<td>0.00011</td>
<td>2.0</td>
<td>0.0378</td>
<td>1.0</td>
<td>0.00784</td>
<td>1.4</td>
<td>0.1066</td>
<td>1.0</td>
</tr>
</tbody>
</table>

In the second set of tests, we use the DLS method (2.4) with polynomial degree \( k = 1, 2, 3, 4 \) again on the hexagon grids shown in Figure 5.3. The error and the order of convergence are listed in Tables 5.5-5.8, where \( Q_h u \) and \( Q_h q \) are the \( L^2 \) projections, and \( u_h \) and \( q_h \) the numerical solutions. The optimal order of convergence is also achieved in all cases. When \( k = 4 \), the exact solution is inside the discontinuous finite element space, and the numerical solution is exact up to the computer accuracy. We
Fig. 5.2. The finite element solutions, $u_h$, $(q_h)_1$ and $(q_h)_2$ of $P_1$ DLS (2.4) for (5.1) on level 4 square grid of Figure 5.1.

Fig. 5.3. The first four hexagon grids for numerical solutions in Tables 5.5-5.8. Plot the finite element solutions on the level 4 hexagon grids in Figure 5.4.

REFERENCES


Table 5.6

The errors, $e_u = Q_h u - u_h$ and $e_q = Q_h q - q_h$, and the order of convergence, by the $P_2$ DLS method (2.4) on hexagon grids (Figure 5.3), for (5.1).

<table>
<thead>
<tr>
<th></th>
<th>$|e_u|_0$</th>
<th>$h^n$</th>
<th>$|e_u|$</th>
<th>$h^n$</th>
<th>$|e_q|_0$</th>
<th>$h^n$</th>
<th>$|e_q|$</th>
<th>$h^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.088752</td>
<td>0.0</td>
<td>0.009023</td>
<td>0.0</td>
<td>2.7502</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.068734</td>
<td>0.4</td>
<td>0.040785</td>
<td>1.2</td>
<td>2.4527</td>
<td>0.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.014699</td>
<td>2.2</td>
<td>0.009128</td>
<td>2.1</td>
<td>0.8278</td>
<td>1.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.002319</td>
<td>2.7</td>
<td>0.02314</td>
<td>2.0</td>
<td>0.2338</td>
<td>1.8</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>2.8</td>
<td>0.00594</td>
<td>2.0</td>
<td>0.0618</td>
<td>1.9</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.000044</td>
<td>2.9</td>
<td>0.00152</td>
<td>2.0</td>
<td>0.0159</td>
<td>2.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.7

The errors, $e_u = Q_h u - u_h$ and $e_q = Q_h q - q_h$, and the order of convergence, by the $P_3$ DLS method (2.4) on hexagon grids (Figure 5.3), for (5.1).

<table>
<thead>
<tr>
<th></th>
<th>$|e_u|_0$</th>
<th>$h^n$</th>
<th>$|e_u|$</th>
<th>$h^n$</th>
<th>$|e_q|_0$</th>
<th>$h^n$</th>
<th>$|e_q|$</th>
<th>$h^n$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.061983</td>
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<td>0.000000</td>
<td>0.0</td>
<td>0.000000</td>
<td>0.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.034715</td>
<td>0.8</td>
<td>0.001122</td>
<td>2.5</td>
<td>0.001180</td>
<td>2.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.002325</td>
<td>3.4</td>
<td>0.001122</td>
<td>2.5</td>
<td>0.001180</td>
<td>2.6</td>
<td></td>
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<tr>
<td>4</td>
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<td>0.001122</td>
<td>2.5</td>
<td>0.001180</td>
<td>2.6</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>4.0</td>
<td>0.000000</td>
<td>0.0</td>
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<td></td>
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<tr>
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<td>4.0</td>
<td>0.000000</td>
<td>0.0</td>
<td>0.000000</td>
<td>0.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.8

The errors and the order of convergence, by the $P_4$ DLS method (2.4) on hexagon grids (Figure 5.3), for (5.1).

<table>
<thead>
<tr>
<th></th>
<th>$|Q_h u - u_h|_0$</th>
<th>$h^n$</th>
<th>$|Q_h u - u_h|_1$</th>
<th>$h^n$</th>
<th>$|Q_h q - q_h|_0$</th>
<th>$h^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0.0000000000</td>
<td>0.0</td>
<td>0.0000000000</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.0000000000</td>
<td>0.0</td>
<td>0.0000000000</td>
<td>0.0</td>
<td>0.0000000000</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>0.0000000000</td>
<td>0.0</td>
<td>0.0000000000</td>
<td>0.0</td>
<td>0.0000000000</td>
<td>0.0</td>
</tr>
</tbody>
</table>

2009.

Fig. 5.4. The finite element solutions, \( u_h, (q_h)_1 \) and \( (q_h)_2 \) of \( P_1 \) DLS (2.4) for (5.1) on the level 4 hexagon grid of Figure 5.3.


