Abstract. Consider the Poisson equation in a polytopal domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) as the model problem. We study interior energy error estimates for the weak Galerkin finite element approximation to elliptic boundary value problems. In particular, we show that the interior error in the energy norm is bounded by three components: the best local approximation error, the error in negative norms, and the trace error on the element boundaries. This implies that the interior convergence rate can be polluted by solution singularities on the domain boundary, even when the solution is smooth in the interior region. Numerical results are reported to support the theoretical findings. To the best of our knowledge, this is the first local energy error analysis that applies to general meshes consisting of polytopal elements and hanging nodes.

Key words. weak Galerkin, finite element methods, interior estimates, second-order elliptic problems

AMS subject classifications. Primary: 65N15, 65N30; Secondary: 35J50

1. Introduction. Let $\Omega$ be a bounded polytopal domain in $\mathbb{R}^d$ for $d = 2, 3$. Namely, $\Omega$ is a polygon ($d = 2$) or a polyhedron ($d = 3$). We consider the Poisson equation with the Dirichlet boundary condition

\[
-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

for a given function $f \in L^2(\Omega)$.

The weak Galerkin (WG) finite element method refers to a general finite element technique solving partial differential equations. The WG methods are designed to use discontinuous approximations on general polytopal meshes. In the presence of discontinuous functions in the finite element approximation, the corresponding discretization formulation is often complicated, and sometimes with undesirable features such as tuning parameters, in order to enforce the interrelation of the finite element solution between elements. In contrast, the WG finite element discretization is featured with a simplified formulation that can be derived directly from the weak form of the corresponding equation by replacing strong derivatives with weakly defined derivatives. First introduced in [28, 29], the WG method has shown its efficacy in a variety of computational models, such as the Stokes problem, transmission problems, and Maxwell’s equations; and its convergence in the global energy norm is well justified (see [21, 22, 20] and references therein).

The interior energy error estimate is a critical component for convergence analysis in other norms and is useful in many applications. For the usual finite element method,
such interior estimates were established on simplicial elements (see for example [27, 19, 23, 26, 24]). It was shown that the interior energy error of the finite element solution is bounded by the best interior energy approximation error from the finite element space and by the error in low-order norms. Different from the best interior approximation error, which depends on the local smoothness of the solution in the interior region, the error in low-order norms is affected by the global smoothness of the solution. On polygonal or polyhedral domains, the error in low-order norms can be the dominant term for the interior energy error, due to the solution singularity near corners and edges of the boundary. This is referred to as the pollution effect of the finite element method. The interior error estimate for the WG method, however, remains an open investigation.

The goal of this paper is to derive local energy error estimates in an interior region of the domain for the WG approximation to the model problem (1.1), which can be applied on general polytopal meshes. Due to the fact that the WG method utilizes a special variational formulation on discontinuous approximation spaces, the well-established local error estimates for the finite element method do not directly apply, and new analysis is needed for further developments. To the best of our knowledge, this is also the first interior energy error analysis that applies to general meshes consisting of polytopal elements and hanging nodes. In addition, we derive negative norm error estimates for the WG method, which can be of independent interest.

In equation (1.1), the solution \( u \in H^1_0(\Omega) \) is unique for \( f \in H^{-1}(\Omega) \). The smoothness of the solution, however, depends on both the given data \( f \) and on the geometry of the domain. In particular, on polytopal domains, it is well known [17, 13, 8, 16, 6, 11, 9] that the solution generally does not satisfy the full regularity estimate (1.2)

\[
\|u\|_{H^{m+1}(\Omega)} \leq C\|f\|_{H^{m-1}(\Omega)}, \quad m \geq 0,
\]

which holds on domains with a smooth boundary. This is because the non-smooth boundary points (corners \( d = 2 \) and vertices and edges \( d = 3 \)) can lead to singularities in the solution. Consequently, even for a sufficiently smooth function \( f \), the solution can merely belong to \( H^{a+1}(\Omega) \), for any \( 0 < a < \eta \), where \( \eta \geq 1/2 \) is the elliptic regularity index determined by the local geometry of the non-smooth boundary points [10]. This lack of regularity can severely slow down the global convergence of the numerical approximation [2, 14, 15, 18, 5, 3, 7, 12, 25] and has been a major concern in the computational community. For the WG approximation, it can be shown that the global convergence rate in the energy norm has an upper bound decided by the regularity of the solution (Proposition 2.2).

Meanwhile, as a local property, the regularity of the solution can vary in different local regions of the domain. Especially, in an interior region, the solution regularity is decided by the local smoothness of the given function. Namely, \( u \in H^{m+1}(G) \) for \( f \in H^{m-1}(G') \) and \( G \subset \subset \Omega \). Thus, the solution may possess more regularity in the interior region. Therefore, an intriguing question that is both of theoretical interest and of practical importance is whether in this case the convergence of the WG solution is better in the interior than in the entire domain. In this paper, by developing interior energy estimates, we shall establish a connection between the interior error and the global approximation error for the WG method. The novelty of the analysis lies in the establishment of a Caccioppoli-type inequality for “discrete harmonic” functions in the discontinuous WG subspace (see (4.3)) and in the derivation of the interior negative norm estimate (Theorem 3.7) and of the estimate (4.15) regarding the bilinear form involving the weak derivatives. To be more precise, through a detailed
study on the approximation properties of the WG method, we show that the interior energy error is bounded by three components: the best interior energy approximation error, the local negative norms of the error, and the local trace error on the element boundaries (Theorem 4.5). Although the last two errors are of higher order compared with the global energy error, they are not necessarily better than the local energy error from the best interior approximation, especially when the domain is non-convex and the solution possesses singularities. Hence, the interior energy approximation of the WG solution can be “polluted” by the singular solution outside of the region, which resembles the local behavior of the standard finite element method [26].

The rest of the paper is organized as follows. In Section 2, we define weak functions and introduce the WG finite element scheme. Then, we describe global approximation properties of the numerical method for possible singular solutions. In Section 3, we study on the approximation properties of the WG method, we show that the interior energy error is bounded by three components: the best interior energy approximation error, the local negative norms of the error, and the local trace error on the element boundaries (Theorem 4.5). Although the last two errors are of higher order compared with the global energy error, they are not necessarily better than the local energy error from the best interior approximation, especially when the domain is non-convex and the solution possesses singularities. Hence, the interior energy approximation of the WG solution can be “polluted” by the singular solution outside of the region, which resembles the local behavior of the standard finite element method [26].

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is said to be in \( H^1(K) \) if it can be identified with a function \( v \in H^1(K) \) using the inclusion map.

Using the notation introduced above, we can formulate a discrete space for the WG finite element approximation. Let \( T_h \) be a partition of the domain \( \Omega \) consisting of shape-regular polygons (\( \Omega \subseteq \mathbb{R}^2 \)) or polyhedra (\( \Omega \subseteq \mathbb{R}^3 \)) satisfying a set of conditions specified in [29]. Denote by \( E_h \) the set of all edges or flat faces in \( T_h \). For every element \( T \in T_h \), we denote by \( h_T \) its diameter and the mesh size is \( h = \max_{T \in T_h} h_T \) for \( T_h \). To simplify the presentation, we assume there exists \( C > 0 \) independent of \( h \), such that \( h \leq C \min_{T \in T_h} h_T \). For a given integer \( k \geq 1 \) and a region \( G \subseteq \Omega \), denote by \( P_k(G) \) the space of polynomials of degree \( \leq k \) on \( G \). Then, we define the space of weak functions associated with \( T_h \),

\[ V = \{ v = (v_0, v_b) : v_0|_T \in H^1(T), v_b|_e \in H^{1/2}(e), \forall e \in E_h, \forall T \in T_h; \ v_b = 0 \text{ on } \partial \Omega \}, \]

and define the WG finite element space associated with \( T_h \),

\[ V_h = \{ v = (v_0, v_b) : v_0|_T \in P_k(T), v_b|_e \in P_k(e), \forall e \in \partial T, \forall T \in T_h; \ v_b = 0 \text{ on } \partial \Omega \}. \]

We emphasize that any \( v \in V_h \) has a single value \( v_b \) on each \( e \in E_h \). In addition, for any \( G \subseteq \Omega \), let \( V_{G,h} \) be the restriction of \( V_h \) on \( G \), and define

\[ \hat{V}_{G,h} = \{ v \in V_{G,h} : \text{supp } v \subseteq G \}. \]

It is clear that \( \hat{V}_{G,h} \subseteq V_h \).

Then, we define the weak gradient for weak functions.

**Definition 2.2.** For \( T \in T_h \) and a weak function \( v = (v_0, v_b) \in W(T) \), a weak gradient \( \nabla_w v \in [P_{k-1}(T)]^d \) on \( T \) is defined by

\[ (\nabla_w v, \tau)_T = -(v_0, \nabla \cdot \tau)_T + (v_b, \tau \cdot n)_{\partial T}, \quad \forall \tau \in [P_{k-1}(T)]^d, \]

where \( (v, w)_T = \int_T vw \, dx, (v, w)_{\partial T} = \int_{\partial T} vw \, ds \), and \( n \) is the outward normal direction to \( \partial T \).

Next, we will introduce three global projections \( Q_0, Q_b, Q_h \). They are elementwise defined \( L^2 \) projections detailed as follows. For each element \( T \in T_h \), \( Q_0 : L^2(T) \rightarrow P_k(T) \) and \( Q_b : L^2(e) \rightarrow P_k(e) \) are the \( L^2 \) projections onto the associated local polynomial spaces, \( Q_h \) is the \( L^2 \) projection from \( [L^2(T)]^d \) to the local weak gradient space \( [P_{k-1}(T)]^d \). Then, we define the projection operator \( Q_h : V \rightarrow V_h \), such that on each element \( T \in T_h \)

\[ Q_h v = (Q_0 v_0, Q_b v_b), \quad \text{for any } v = (v_0, v_b) \in V. \]

If a function \( v = (v_0, v_b) \) is smoother such that for any \( e \in E_h \), \( v_0|_e = v_b|_e \), we simplify the notation for \( Q_h v \) by \( Q_h v = (Q_0 v, Q_b v) \).

Note that \( Q_h \) and \( Q_b \) are locally defined operators. We further assume the partition \( T_h \) gives rise to the optimal local convergence rates (2.3) – (2.5) and the standard inverse inequality (2.6). Namely, for \( 1/2 < a \leq k \), the following approximation properties hold

\[ \| v - Q_0 v \|_{0,T} + h_T \| \nabla (v - Q_0 v) \|_{0,T} \leq C h_T^{a+1} \| v \|_{a+1,T}, \quad \forall v \in H^{a+1}(T), \]

\[ \| v - Q_b v \|_{0,e} \leq C h_T^{a+1} \| v \|_{a+1,e}, \quad \forall v \in H^{a+1}(e), \]
and for $0 \leq a_1 \leq 1$ and $v \in [H^{a_2}(T)]^d$, where $a_1 < a_2 \leq k$, we have
\begin{equation}
\|v - Q_h v\|_{a_1,T} \leq Ch_T^{a_2-a_1} \|v\|_{a_2,T}.
\end{equation}

In addition, for $v \in V_h$,
\begin{equation}
|v|_{l,T} \leq Ch_T^{-l} \|v\|_{0,T}, \quad 0 \leq l \leq k.
\end{equation}

Examples of such partitions include shape-regular triangular or tetrahedral triangulations, and polytopal partitions in [22, 29]. Note that the estimate (2.5) is a consequence of the operator-interpolation theory (Chapter 14 in [4]) and the approximation property (Lemma 4.1 in [29]).

\|v - Q_h v\|_{0,T} + h_T |v - Q_h v|_{1,T} \leq C h_T^k \|v\|_{k,T}.

### 2.2. Weak Galerkin finite element schemes

We introduce necessary bilinear forms in order to formulate the numerical scheme. For $v, w \in V$, define
\begin{align*}
s_T(v, w) &= h_T^{-1} \langle v_0 - v_b, w_0 - w_b \rangle_{\partial T}, \\
a_T(v, w) &= (\nabla_w v, \nabla_w w)_T + s_T(v, w);
\end{align*}

and
\begin{align}
(2.7) &\quad s(v, w) = \sum_{T \in T_h} s_T(v, w), \quad a(v, w) = \sum_{T \in T_h} a_T(v, w).
\end{align}

For $G \subseteq \Omega$, we denote by $s_G(\cdot, \cdot)$ the restriction of $s(\cdot, \cdot)$ on $G$.

Then, we define the WG approximation of equation (1.1) as in [21].

**Weak Galerkin Algorithm 1.** A unique numerical approximation for (1.1) can be obtained by seeking $u_h = \{u_0, u_b\} \in V_h$ that satisfies the following equation:
\begin{equation}
a(u_h, v) = (f, v_0), \quad \forall v = \{v_0, v_b\} \in V_h.
\end{equation}

For $v \in V$, we define a norm associated to the bilinear form
\begin{equation}
\|v\|_{\Omega} = \sqrt{a(v, v)}.
\end{equation}

In addition, for $v \in V_h + H^1_0(\Omega)$, define the discrete $H^1$ energy norm
\begin{equation}
\|v\|_{h,\Omega} = \left( \sum_{T \in T_h} \left( \|\nabla v_0\|_{0,T}^2 + h_T^{-1} \|v_0 - v_b\|_{0,\partial T}^2 \right) \right)^{1/2}.
\end{equation}

In particular, if $v \in H^1(\Omega)$, it can be seen that $\|v\|_{h,\Omega} = \|\nabla v\|_{0,\Omega}$. For a subregion $G \subseteq \Omega$, we denote by $\|\cdot\|_{h,G}$ and $\|\cdot\|_G$ the restrictions of $\|\cdot\|_{h,\Omega}$ and $\|\cdot\|_{\Omega}$ on $G$, respectively. Moreover, the following discrete Poincaré inequality holds for functions in $V_h$ (Lemma 3.4 in [20]).

**Proposition 2.1.** For $v = \{v_0, v_b\} \in V_h$, we have
\begin{equation}
\sum_{T \in T_h} \|v_0\|_{0,T}^2 + h_T \|v_0 - v_b\|_{0,T}^2 \leq C \|\nabla_w v\|_{0,\Omega}^2.
\end{equation}
Before proceeding with our interior analysis, we recall the global error estimates regarding the WG finite element method.

**Proposition 2.2.** Let \( e_h = Q_h u - u_h \), where \( u_h = \{u_0, u_b\} \) is the WG finite element solution arising from (2.8). Then, for any \( v \in V_h \) we have

\[
a(e_h, v) = \ell_u(v) + s(Q_h u, v),
\]

where \( \ell_u(v) = \sum_{T \in T_h} \langle (\nabla u - Q_h \nabla u) \cdot n, v_0 - v_b \rangle_{\partial T} \). Consequently, if the exact solution \( u \in H^{\mu+1}(\Omega) \) for \( \mu > 0 \), we have

\[
\|u - u_h\|_{h,\Omega} \leq \|u - Q_h u\|_{h,\Omega} + \|u_h - Q_h u\|_{\Omega} \leq Ch^{t}\|u\|_{t+1,\Omega},
\]

where \( t = \min(\mu, k) \). In addition, for any \( v \in V_h \), let \( G \) be a region such that \( \text{supp } v \subseteq G \subseteq \Omega \). Using the same notation \( \mu \) and \( t \), we have

\[
|s(Q_h u, v)| \leq Ch^{1}s_{G}(v, v)^{1/2}\|u\|_{t+1,G} \leq Ch^{t}\|u\|_{t+1,G}\|v\|_{G},
\]

\[
|\ell_u(v)| \leq Ch^{t}\|v\|_{t+1,G}\|v\|_{G}.
\]

**Proof.** The error equation (2.11) can be found in [21]. The estimates (2.13) and (2.14) follow from the approximation properties (2.3) – (2.5) and from the same lines as in Lemma 5.4, [22]. The energy norm estimate (2.12) can be derived using the error equation (2.11) and the estimates in (2.13) and (2.14). \( \square \)

**Remark 2.3.** The global estimates in Proposition 2.2 were derived based on the local approximation properties associated to the \( L^2 \) projection operators \( Q_h \) and \( Q_h \) for the solution \( u \in H^{\mu+1}(\Omega) \). It is well known that the solution regularity in equation (1.1) depends on the given function \( f \) and on the geometry of the domain. For example, on a polytopal domain, the solution can possess singularities near non-smooth points on the boundary, and the full regularity estimate (1.2) does not hold for all \( m \geq 0 \). In fact, there is a regularity index \( \eta \geq 1/2 \) that is determined by the domain geometry, such that \( u \) only belongs to \( H^{a+1}(\Omega) \) for \( 0 < a < \eta \), even when the given data is smooth (see Chapter 2 in [10]), and it holds that

\[
\|u\|_{a+1,\Omega} \leq C\|f\|_{a-1,\Omega}.
\]

In particular, on a polygon, \( \eta = \pi/\chi \), where \( \chi \) is the largest interior angle of the domain.

Meanwhile, the regularity is a local property. In an interior region \( G \subset \subset \Omega \), the solution regularity is determined by the given function \( f \) (see [10]), since it is away from the boundary. Therefore, the solution of equation (1.1) may possess different smoothness in \( G \) than other parts of the domain. Our study shall shed light on local energy error analysis of the WG method that applies to general meshes consisting of polytopal elements and hanging nodes.

3. Negative norm estimates. In this section, we derive global and interior negative norm estimates for the WG method.

3.1. Some lemmas. Let \( T \in T_h \) be an element and let \( e \in E_h \) be such that \( e \subset \partial T \). We first recall the trace inequality [29] for a function \( \varphi \in H^1(T) \),

\[
\|\varphi\|_{e}^2 \leq C \left( h_T^{-1}\|\varphi\|_{T}^2 + h_T\|\nabla \varphi\|_{T}^2 \right).
\]
For functions in $V_h$, the gradient operator and the weak gradient operator are connected through (2.2), and satisfy the following estimates.

**Lemma 3.1.** For $v \in V_h$ and $T \in T_h$, we have

\[
\|\nabla v_0\|_T \leq C \left(\|\nabla w v\|_T + h^{-1/2}_T \|v_0 - v_b\|_{\partial T}\right),
\]

(3.2)

\[
\|\nabla w v\|_T \leq C \left(\|\nabla v_0\|_T + h^{-1/2}_T \|v_0 - v_b\|_{\partial T}\right).
\]

(3.3)

**Proof.** Let $\tau = \nabla v_0$ in (2.2). Using integration by parts, we obtain

\[
(\nabla w v, \nabla v_0)_T = (\nabla v_0, \nabla v_0)_T + (v_b - v_0, \nabla v_0 \cdot n)_{\partial T}.
\]

Using the trace estimate (3.1) and the inverse inequality, we have

\[
\|\nabla v_0\|^2_T \leq \|\nabla v_0\|_T \|\nabla v_0\|_T + \|v_0 - v_b\|_{\partial T} \|\nabla v_0\|_{\partial T}
\]

\[
\leq \|\nabla v_0\|_T \|\nabla v_0\|_T + h^{-1/2}_T h^{1/2}_T (h^{-1/2}_T \|\nabla v_0\|_T + h^{1/2}_T \|\nabla v_0\|_{1,T})
\]

\[
= \|\nabla v_0\|_T \|\nabla v_0\|_T + Ch^{-1/2}_T \|v_0 - v_b\|_{\partial T} \|\nabla v_0\|_{T},
\]

which leads to

\[
\|\nabla v_0\|_T \leq C \left(\|\nabla v_0\|_T + h^{-1/2}_T \|v_0 - v_b\|_{\partial T}\right).
\]

The estimate (3.3) follows from a similar calculation. \(\square\)

In addition, the $L^2$ projection $Q_h$ possesses the following local properties.

**Lemma 3.2.** Let $T \in T_h$ and $v \in H^1(T)$. Then, we have

\[
\|v - Q_h v\|_{0,\partial T} \leq \|v - Q_0 v\|_{0,\partial T}.
\]

(3.4)

Let $\omega \in C^\infty(\bar{T})$ such that $|D^l \omega| < C_0$ for $0 \leq l \leq k + 1$, where $C_0$ is a constant independent of $h$. Then, for $e \in E_h$ with $e \subset \partial T$, we have

\[
h^{-1}_T \|\omega v - Q_0(\omega v)\|_{0,T} + \|\nabla(\omega v - Q_0(\omega v))\|_{0,T} \leq C_h T \|v\|_{1,T}, \quad \forall v \in P_k(T),
\]

(3.5)

\[
\|\omega v - Q_0(\omega v)\|_{0,e} \leq C_h T \|v\|_{0,e}, \quad \forall v \in P_k(e).
\]

(3.6)

**Proof.** The estimate (3.4) follows from the following calculation

\[
\|v - Q_h v\|^2_{0,\partial T} = \langle v - Q_h v, v - Q_h v\rangle_{\partial T}
\]

\[
= \langle v - Q_h v, v - Q_0 v\rangle_{\partial T} \leq \|v - Q_h v\|_{0,\partial T} \|v - Q_0 v\|_{0,\partial T}.
\]

For $v \in P_k(T)$ and $|\alpha| = k + 1$, note that $\partial^\alpha v = 0$ on $T$. Recall the approximation property in (2.3). Then, using the inverse inequality, we have

\[
h^{-2}_T \|\omega v - Q_0(\omega v)\|^2_{0,T} + \|\nabla(\omega v - Q_0(\omega v))\|^2_{0,T} \leq Ch^{2k}_T \|\omega v\|^2_{k+1,T}
\]

\[
\leq Ch^{2k}_T \sum_{0 \leq i \leq k+1} \|\omega v\|^2_{i,T} \leq Ch^{2k}_T \sum_{0 \leq i \leq k} \|v\|^2_{i,T} \leq Ch^{2k}_T \|v\|^2_{1,T},
\]

which proves the estimate (3.5). The estimate (3.6) follows from a similar calculation based on the approximation property (2.4). \(\square\)

The estimates (3.5) and (3.6) do not include high-order norms on the right hand side and represent certain super-approximation properties for the functions involved.
In addition, we have a useful property regarding the $L^2$ projection operators $Q_h$ and $Q_h^l$ in Definition 2.2.

**Lemma 3.3.** For each $T \in \mathcal{T}_h$ and $v \in H^1(T)$, we have

$$
\nabla_w(Q_h v) = Q_h(\nabla v).
$$

**Proof.** Following the definition in (2.2), we have for any $\tau \in [P_{k-1}(T)]^d$,

$$(\nabla_w(Q_h v), \tau)_T = -(Q_0 v, \nabla \cdot \tau)_T + (Q_0 v, \tau \cdot n)_{\partial T} = -(v, \nabla \cdot \tau)_T + (v, \tau \cdot n)_{\partial T} = (\nabla v, \tau)_T = (Q_h(\nabla v), \tau)_T,$$

which completes the proof. \( \square \)

### 3.2. Global negative norm analysis

Consider the auxiliary problem

$$
-\Delta \phi = g \quad \text{in} \quad \Omega, \quad \phi = 0 \quad \text{on} \quad \partial \Omega,
$$

where $g \in C_0^\infty(\Omega)$. Therefore, according to Remark 2.3,

$$
\|\phi\|_{a+1,\Omega} \leq C\|g\|_{a-1,\Omega}, \quad \forall 0 \leq a < \eta,
$$

where $\eta > 1/2$ is the regularity index. Then, using a duality argument, we obtain the global negative norm estimate for the WG solution.

**Theorem 3.4.** Suppose the solution of equation (1.1) $u \in H^{\mu+1}(\Omega)$, for some $\mu > 1/2$. Let $t = \min(\mu, k)$ and $p = \min(a, k)$, where $a$ is the index in (3.8). Then, for $l \geq 0$, the WG solution $u_h = \{u_0, u_b\}$ satisfies

$$
\|u - u_0\|_{-l,\Omega} \leq \begin{cases} 
Ch^{l+1} & \text{if } 0 \leq l \leq p - 1, \\
Ch^{l+p} & \text{if } l > p - 1.
\end{cases}
$$

**Proof.** Let $e_0 = Q_0 u - u_0$, $e_b = Q_b u - u_b$, and $e_h = \{e_0, e_b\}$. Note that

$$
\|u - u_0\|_{-l,\Omega} \leq \|u - Q_0 u\|_{-l,\Omega} + \|Q_0 u - u_0\|_{-l,\Omega}.
$$

Since $Q_0$ is the local $L^2$ projection on each element, we have

$$
norm{u - Q_0 u}_{-l,\Omega} = \sup_{g \neq 0 \in C_0^\infty(\Omega)} \frac{(u - Q_0 u, g)}{\|g\|_{l,\Omega}} = \sup_{g \neq 0 \in C_0^\infty(\Omega)} \frac{(u - Q_0 u, g - Q_0 g)}{\|g\|_{l,\Omega}}
\leq Ch^{l+1} \|u\|_{l+1,\Omega} \quad \text{if } 0 \leq l \leq k + 1,
$$
or

$$
norm{u - Q_0 u}_{-l,\Omega} \leq Ch^{l+k+2} \|u\|_{l+1,\Omega} \quad \text{if } l > k + 1.
$$

For the second term in (3.10), using (3.7), we obtain

$$
norm{e_0}_{-l,\Omega} = \sup_{g \neq 0 \in C_0^\infty(\Omega)} \frac{(e_0, g)}{\|g\|_{l,\Omega}} = \sup_{g \neq 0 \in C_0^\infty(\Omega)} \frac{(e_0, -\Delta \phi)}{\|g\|_{l,\Omega}}.
$$

Using integration by parts and $e_b|_{\partial \Omega} = 0$, we have

$$
(e_0, -\Delta \phi) = \sum_{T \in \mathcal{T}_h} \langle \nabla \phi, \nabla e_0 \rangle_T - \sum_{T \in \mathcal{T}_h} \langle \nabla \phi \cdot n, e_0 - e_b \rangle_{\partial T}.
$$
With the estimate in Lemma 3.3, (2.2), and integration by parts, it holds that

\[
(\nabla_w Q_h \phi, \nabla_w e_h)_T = (Q_h(\nabla \phi)\cdot \nabla e_h)_T \\
= -(e_0, \nabla \cdot (Q_h \nabla \phi))_T + \langle e_b, Q_h(\nabla \phi) \cdot n \rangle_{\partial T} \\
= (\nabla e_0, Q_h \nabla \phi)_T - \langle e_0 - e_b, Q_h(\nabla \phi) \cdot n \rangle_{\partial T} \\
= (\nabla \phi, \nabla e_0)_T - \langle e_0 - e_b, Q_h(\nabla \phi) \cdot n \rangle_{\partial T}.
\]

(3.14)

Thus, by (3.13), (3.14), and (2.11), we have

\[
(e_0, -\Delta \phi) = \sum_{T \in T_h} ((\nabla_w Q_h \phi, \nabla_w e_h)_T + \langle (Q_h(\nabla \phi) - \nabla \phi) \cdot n, e_0 - e_b \rangle_{\partial T}) \\
= \ell_u(Q_h \phi) + s(Q_h u, Q_h \phi) - s(e_b, Q_h \phi) - \ell_\phi(e_h),
\]

where \( \ell_u(Q_h \phi) \) is defined as in Proposition 2.2. Then, by the triangle inequality,

\[
|\ell_u(Q_h \phi)| \leq \sum_{T \in T_h} |((\nabla u - Q_h \nabla u) \cdot n, Q_0 \phi - \phi)_{\partial T}| \\
+ |\sum_{T \in T_h} ((\nabla u - Q_h \nabla u) \cdot n, Q_b \phi - \phi)_{\partial T}|.
\]

(3.16)

Note that \( Q_b \phi_{\partial T} = \phi_{\partial T} = 0 \). Therefore,

\[
\sum_{T \in T_h} ((\nabla u - Q_h \nabla u) \cdot n, Q_b \phi - \phi)_{\partial T} = \sum_{T \in T_h} (\nabla u \cdot n, Q_b \phi - \phi)_{\partial T} = 0.
\]

Meanwhile, using the trace estimate (3.1), we have

\[
\|Q_0 \phi - \phi\|_{0, \partial T} \leq C(h_T^{-1/2})\|Q_0 \phi - \phi\|_{0,T} + h_T^{1/2}\|\nabla(Q_0 \phi - \phi)\|_{0,T}.
\]

Therefore, using the trace estimate, (2.3), and (2.5), we have

\[
\|Q_0 \phi - \phi\|_{0, \partial T} \leq C(h_T^{3/2+l})\|\phi\|_{l+2,T} \text{ for } 0 \leq l \leq p - 1,
\]

(3.17)

\[
\|Q_0 \phi - \phi\|_{0, \partial T} \leq C(h_T^{1/2+p})\|\phi\|_{p+1,T} \text{ for } l > p - 1,
\]

(3.18)

\[
\|\nabla u - Q_h \nabla u\|_{0, \partial T} \leq C(h_T^{-1/2})\|u\|_{l+1,T}.
\]

(3.19)

We briefly show the estimate (3.19). Suppose a vertex of \( T \in T_h \) lies at the origin. Let \( \hat{T} = h_T^{-1} \circ T \) be the reference polytopal obtained after the scaling with factor \( h_T^{-1} \) at the origin. According to the trace estimate [10], for \( v \in H^{1/2+\epsilon}(\hat{T}) \) with \( \epsilon > 0 \) arbitrarily small, \( \|v\|_{L^2(\hat{T})} \leq C(\|v\|_{L^2(T)} + |v|_{H^{1/2,\epsilon}(T)}) \), where \( |v|_{H^{1/2,\epsilon}(T)} = \left( \int_T \int_T \frac{|v(x) - v(y)|^2}{|x-y|^{d+1+\epsilon}} dx dy \right)^{1/2} \) is the \( H^{1/2,\epsilon}(\hat{T}) \) semi-norm, and \( e \in E_h \) for any \( e \in T_h \) and \( e \subset \partial T \). Using the scaling argument, we have \( \|v\|_{L^2(\hat{T})} \leq C(h_T^{-1/2})\|v\|_{L^2(T)} + h_T^{\epsilon}|v|_{H^{1/2+\epsilon}(T)} \leq C(h_T^{-1/2})\|v\|_{L^2(T)} + h_T^{-1/2}|v|_{H^{1/2+\epsilon}(T)} \). Recall \( t > 1/2 \). Replacing \( v \) by \( \nabla u - Q_h \nabla u \) and using the estimate in (2.5), we obtain the inequality in (3.19).

Therefore, by (3.16) – (3.19) and the Cauchy-Schwarz inequality, it follows that

\[
|\ell_u(Q_h \phi)| \leq C \left( \sum_{T \in T_h} \|\nabla u - Q_h \nabla u\|_{0, \partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} \|Q_0 \phi - \phi\|_{0, \partial T}^2 \right)^{1/2} \\
\leq C(LT)h^T\|u\|_{l+1, \Omega}.
\]

(3.20)
where

\begin{equation}
LT := \begin{cases} 
h^{l+1}||\phi||_{l+2,\Omega} & \text{if } 0 \leq l \leq p - 1, \\
h^p \|\phi\|_{p+1,\Omega} & \text{if } l > p - 1.
\end{cases}
\end{equation}

Similarly, by the definition of $Q_b$, the trace estimate (3.1), and (2.3), we have

\begin{align}
|s(Q_h u, Q_h \phi)| &\leq \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_b(Q_0 u - u)\|_{0,\partial T} \|Q_b(Q_0 \phi - \phi)\|_{0,\partial T} \\
&\leq \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 u - u\|_{0,\partial T} \|Q_b \phi - \phi\|_{0,\partial T} \\
&\leq C \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 u - u\|_{0,\partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_b \phi - \phi\|_{0,\partial T}^2 \right)^{1/2} \\
&\leq C(LT) h^t \|u\|_{t+1,\Omega}.
\end{align}

(3.22)

For the third term in (3.15), using a similar calculation as above, by (3.3), the estimate in Proposition 2.2, (2.3), and (2.4), we have

\begin{align}
|s(e_h, Q_h \phi)| &\leq C \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_b \phi - \phi\|_{0,\partial T}^2 \right)^{1/2} \|e_h\|_{\Omega} \\
&\leq C(LT) \left( \|u - u_h\|_{h,\Omega} + \|u - Q_h u\|_{h,\Omega} \right) \leq C(LT) h^t \|u\|_{t+1,\Omega}.
\end{align}

(3.23)

For the last term in (3.15), by the trace estimate, Proposition 2.2, and (3.19), we have

\begin{align}
\|\phi(e_h)\| = |\sum_{T \in \mathcal{T}_h} (\langle Q_h (\nabla \phi) - \nabla \phi\rangle \cdot n, e_0 - e_h)_{\partial T}| \\
&\leq C \left( \sum_{T \in \mathcal{T}_h} \|Q_h (\nabla \phi) - \nabla \phi\|_{0,\partial T}^2 \right)^{1/2} h^{1/2} \|e_h\|_{\Omega} \leq C(LT) h^t \|u\|_{t+1,\Omega}.
\end{align}

(3.24)

Then, by (3.15) and (3.20) – (3.24), we obtain

\begin{align}
\|(e_0 - \Delta \phi)\| &\leq \begin{cases} 
h^{l+1} \|u\|_{l+2,\Omega} \|\phi\|_{l+2,\Omega} & \text{if } 0 \leq l \leq p - 1, \\
h^p \|u\|_{l+1,\Omega} \|\phi\|_{p+1,\Omega} & \text{if } l > p - 1.
\end{cases}
\end{align}

(3.25)

Therefore, according to (3.8), we obtain

\begin{align}
\|\phi\|_{l+2,\Omega} &\leq C \|g\|_{l,\Omega} \quad \text{if } 0 \leq l \leq p - 1, \\
\|\phi\|_{p+1,\Omega} &\leq C \|g\|_{p-1,\Omega} \leq C \|g\|_{l,\Omega} \quad \text{if } l > p - 1.
\end{align}

(3.26) \quad (3.27)

The proof is completed by combining (3.10) – (3.12) and (3.25) – (3.27). \qed

### 3.3. Interior negative norm analysis.

We start to analyze the negative norm of the error in an interior region. To fix the notation, we carry out the analysis on $d$-dimensional concentric balls $D_i$, $0 \leq i \leq 3$, such that $D_0 \subset D_1 \subset D_2 \subset D_3 \subset \Omega$. In the text below, we assume the mesh parameter $h$ is sufficiently small, such that for $0 \leq i \leq 2$, $\text{dist}(\partial D_i, \partial D_{i+1}) \geq k_0 h$, where $k_0$ is a positive constant.

For $G \subset \Omega$, let $\mathcal{T}_G = \{T_i\}$ be the set of elements such that $T_i \in \mathcal{T}_h$ and $T_i \subset G$. Denote by $\mathcal{G} = \cup_{T \in \mathcal{T}_G} T$. Therefore, for any $v \in V_{G,h}$ (see (2.1)), $v|_{\Omega \setminus \mathcal{G}} = 0$. To study
the local error, we introduce the following cutoff smooth functions. For $1 \leq i \leq 3$, let $\omega_i \in C^\infty_0(D_i)$ be such that

$$\omega_i = 1 \quad \text{in } \hat{D}_i.$$  \hfill (3.28)

Then, when $h$ is sufficiently small, we have $D_i \subseteq \hat{D}_{i+1} \subseteq D_{i+1}$, $0 \leq i \leq 2$.

Let $S(\omega_2 u) = [S(\omega_2 u)_0, [S(\omega_2 u)]_b] \in \mathcal{V}_{D_3, h}$ be the WG approximation of $\omega_2 u$ on $\hat{D}_3$, namely

$$a(S(\omega_2 u), v) = (-\Delta(\omega_2 u), v_0), \quad \forall v \in \hat{V}_{D_3, h}.$$  \hfill (3.29)

In what follows, we redefine the auxiliary equation (3.7), such that $\Omega = D_2$ and $g \in C^\infty_0(D_2)$. Since the ball $D_2$ has a smooth boundary, it holds for any $l \geq 0$ that

$$\|\phi\|_{l+2, D_2} \leq C\|g\|_{l, D_2}.$$  \hfill (3.30)

Recall the WG solution of equation (1.1) $u_h = \{u_0, u_0\}$. Then, we derive the following equation.

**Lemma 3.5.** Let $v_0 = u - u_0$. Then, for $h$ sufficiently small, we have

$$(\omega_1 v_0, -\Delta \phi)_{D_2} = (v_0, \nabla \cdot (\phi \nabla \omega_1))_{D_2} + (\nabla \phi, v_0 \nabla \omega_1)_{D_2} + (v_0, -\Delta (\omega_1 \phi))_{D_2}.$$  \hfill (3.31)

**Proof.** Using $\text{supp}(\omega_1) \subset \hat{D}_2$ and integration by parts, we have

$$(\omega_1 v_0, -\Delta \phi)_{D_2} = -\sum_{T \in \hat{T}_{D_2}} \langle \omega_1 v_0, \nabla \phi \cdot n \rangle_{\partial T} + \sum_{T \in \hat{T}_{D_2}} \langle \nabla \phi, \omega_1 \nabla v_0 + v_0 \nabla \omega_1 \rangle_T$$

$$= \sum_{T \in \hat{T}_{D_2}} (\langle \nabla (\omega_1 \phi), \nabla v_0 \rangle_T - \langle \phi \nabla \omega_1, \nabla v_0 \rangle_T) + (\langle \nabla \phi, v_0 \nabla \omega_1 \rangle_T - \langle \omega_1 v_0, \nabla \phi \cdot n \rangle_{\partial T})$$

$$= \sum_{T \in \hat{T}_{D_2}} (\langle \nabla (\omega_1 \phi), \nabla v_0 \rangle_T + \langle v_0, \nabla \cdot (\phi \nabla \omega_1) \rangle_T + \langle \nabla \phi, v_0 \nabla \omega_1 \rangle_T - \langle v_0, \nabla (\omega_1 \phi) \cdot n \rangle_{\partial T})$$

$$= (v_0, \nabla \cdot (\phi \nabla \omega_1))_{D_2} + (\nabla \phi, v_0 \nabla \omega_1)_{D_2} + (v_0, -\Delta (\omega_1 \phi))_{D_2},$$

which completes the proof. \hfill \Box

Next, we derive an estimate regarding $Q_0(\omega_2 u) - [S(\omega_2 u)]_0$.

**Lemma 3.6.** Define $q_h = \{q_0, q_h\} = Q_h(\omega_2 u) - S(\omega_2 u) \in V_h$. Suppose $u \in H^{\mu+1}(D_2)$ for some $\mu > 1/2$. Let $t = \min(\mu, k)$. For $l \geq 0$, let $l_k = \min(l, k - 1)$. Then, for $h$ sufficiently small, we have

$$|(q_0, -\Delta (\omega_1 \phi))_{D_2}| \leq C h^{t+k+1} \|u\|_{t+1, D_2} \|\phi\|_{t+2, D_1}.$$  \hfill (3.31)

**Proof.** Using integration by parts and noting $\omega_1|_{\partial D_2} = 0$, we have

$$(q_0, -\Delta (\omega_1 \phi))_{D_2} = \sum_{T \in \hat{T}_{D_2}} \langle \nabla (\omega_1 \phi), \nabla q_0 \rangle_T - \sum_{T \in \hat{T}_{D_2}} \langle \nabla (\omega_1 \phi) \cdot n, q_0 - q_h \rangle_{\partial T}.$$  \hfill (3.32)

Replacing $\phi$ by $\omega_1 \phi$ and $e_h$ by $q_h$ in (3.14), we have

$$(\nabla_w Q_h(\omega_1 \phi), \nabla_w q_h)_T = (\nabla (\omega_1 \phi), \nabla q_0)_T - (q_0 - q_h, Q_h(\nabla (\omega_1 \phi)) \cdot n)_{\partial T}.$$
Using (3.32), (3.31), (3.29), and (2.11), we further obtain
\[
(q_0, -\Delta (\omega_1 \phi) = (\nabla_w Q_h(\omega_1 \phi), \nabla_w q_h)_{D_2} + \sum_{T \in T_{D_2}} \langle (Q_h(\nabla (\omega_1 \phi)) - \nabla (\omega_1 \phi)) \cdot n, q_0 - q_h \rangle_{\partial T}
\]
\[
= (\nabla_w Q_h(\omega_1 \phi), \nabla_w q_h)_{D_3} + \sum_{T \in T_{D_3}} \langle (Q_h(\nabla (\omega_1 \phi)) - \nabla (\omega_1 \phi)) \cdot n, q_0 - q_h \rangle_{\partial T}
\]
\[
(3.33) \quad = \ell_{\omega_2 u}(Q_h(\omega_1 \phi))|_{\partial D_3} + s_{D_3}(Q_h(\omega_2 u), Q_h(\omega_1 \phi)) - s_{D_3}(q_h, Q_h(\omega_1 \phi)) - \ell_{\omega_1 \phi}(q_h)|_{\partial D_3}
\]
Recall \( \phi \in H^{l+2}(D_2) \) for any \( l \geq 0 \) (see (3.30)). Replacing \( u \) by \( \omega_2 u \) and \( \phi \) by \( \omega_1 \phi \) in (3.16), (3.20), and noting that \( Q_h(\omega_1 \phi)|_{\partial D_3} = 0 \), we have
\[
|\ell_{\omega_2 u}(Q_h(\omega_1 \phi))|_{D_3} \leq C h^{l+1+k+1} ||\omega_2 u||_{t+1,D_2} ||\omega_1 \phi||_{l+2,D_3}
\]
\[
(3.34) \quad \leq C h^{l+1+k+1} ||u||_{t+1,D_2} ||\phi||_{l+2,D_1}.
\]
Similarly, replacing \( u \) by \( \omega_2 u \) and \( \phi \) by \( \omega_1 \phi \) in (3.22), we have
\[
|s_{D_3}(Q_h(\omega_2 u), Q_h(\omega_1 \phi))| \leq C h^{l+1+k+1} ||\omega_2 u||_{t+1,D_2} ||\omega_1 \phi||_{l+2,D_3}
\]
\[
(3.35) \quad \leq C h^{l+1+k+1} ||u||_{t+1,D_2} ||\phi||_{l+2,D_1}.
\]
Replacing \( u \) by \( \omega_2 u \), \( \phi \) by \( \omega_1 \phi \), and \( e_h \) by \( q_h \) in (3.23), it follows that
\[
|s_{D_3}(q_h, Q_h(\omega_1 \phi))| \leq C h^{l+1+k+1} ||u||_{t+1,D_2} ||\phi||_{l+2,D_1}.
\]
Replacing \( u \) by \( \omega_2 u \), \( \phi \) by \( \omega_1 \phi \), and \( e_h \) by \( q_h \) in (3.24), we obtain
\[
|\ell_{\omega_1 \phi}(q_h)|_{D_3} \leq C h^{l+1+k+1} ||u||_{t+1,D_2} ||\phi||_{l+2,D_1}.
\]
Hence, the proof is completed by combining the estimates (3.33) - (3.37).

Then, we present our interior negative norm estimate for the WG approximation.

**Theorem 3.7.** Recall the concentric balls \( D_0 \subseteq D_1 \subseteq D_2 \subseteq D_3 \subseteq \Omega \). Suppose the solution of equation (1.1) \( u \in H^{l+1}(D_2) \) for some \( l \geq 1/2 \). Let \( t = \min(\mu, k) \). For \( l \geq 0 \), let \( l_k = \min(l, k-1) \). Then, for \( h \) sufficiently small, we have
\[
||u - u_0||_{l,D_0} \leq C h^{l+1+k+1} ||u||_{t+1,D_2} + h^{l+k+1} s_{D_2}(u_h, u_h)^{1/2} + ||u - u_0||_{l-1,D_2},
\]
where \( u_h = \{u_0, u_h\} \) is the WG solution defined in (2.8).

**Proof.** Recall the redefined auxiliary equation \( -\Delta \phi = g \in C_0^\infty(D_2) \) with the Dirichlet boundary condition and its regularity estimate (3.30). Then, by the definitions of negative norms and the cutoff function \( \omega_1 \), we have
\[
||u - u_0||_{l,D_0} = ||\omega_1(u - u_0)||_{l,D_0} \leq ||\omega_1(u - u_0)||_{l-1,D_2}
\]
\[
(3.38) \quad = \sup_{g \neq 0 \in C_0^\infty(D_2)} \frac{(\omega_1(u - u_0), g)_{D_2}}{||g||_{l,D_2}} = \sup_{g \neq 0 \in C_0^\infty(D_2)} \frac{(\omega_1(u - u_0), -\Delta \phi)_{D_2}}{||g||_{l,D_2}}.
\]
Let \( v_0 = u - u_0 \). By Lemma 3.5 and the boundedness of \( \omega_1 \), we have
\[
(\omega_1 v_0, -\Delta \phi)_{D_2} = (v_0, \nabla \cdot (\phi \nabla \omega_1))_{D_2} + (\nabla \phi, v_0 \nabla \omega_1)_{D_2} + (v_0, -\Delta (\omega_1 \phi))_{D_2}
\]
\[
(3.39) \quad \leq C ||v_0||_{l-1,D_2} ||\phi||_{l+2,D_2} + (v_0, -\Delta (\omega_1 \phi))_{D_2}.
\]
We write

\[(3.40) \quad v_0 = u - Q_0(\omega_2 u) + Q_0(\omega_2) - [S(\omega_2)]_0 + [S(\omega_2)]_0 - u_0.\]

In view of (3.39), we shall analyze these terms separately.

By the definition of $Q_0$, $\text{supp}(\omega_1) \subseteq D_1$, and the boundedness of $\omega_2$, we have

\[(u - Q_0(\omega_2 u), -\Delta(\omega_1))_{D_2} = (\omega_2 u - Q_0(\omega_2 u), -\Delta(\omega_1))_{D_2} = (\omega_2 u - Q_0(\omega_2 u), -\Delta(\omega_1) + \Delta(\omega_1))_{D_2} \leq Ch^{l+1}||u||_{l+1,D_2}||\omega||_{l+2,D_1}.\]

(3.41)

Let $w_h = S(\omega_2 u) - u_h = \{w_0, w_h\}$. Note that $\omega_1|_{\partial D_2} = 0$. Therefore, using integration by parts and equation (3.14), we have

\[(w_0, -\Delta(\omega_1))_{D_2} = \sum_{T \in T_{D_2}} ((\nabla(\omega_1), \nabla w_0)_{\mathcal{T}} - (w_0, \nabla(\omega_1) \cdot n)_{\partial T})
= \sum_{T \in T_{D_2}} ((\nabla(\omega_1), \nabla w_0)_{\mathcal{T}} - (w_0 - w_b, \nabla(\omega_1) \cdot n)_{\partial T})
= \sum_{T \in T_{D_2}} ((\nabla(\omega_1), \nabla w_0)_{\mathcal{T}} - (w_0 - w_b, Q_h(\nabla(\omega_1)) \cdot n)_{\partial T})
- (w_0 - w_b, [\nabla(\omega_1) - Q_h(\nabla(\omega_1))] \cdot n)_{\partial T})
= \sum_{T \in T_{D_2}} ((\nabla w(Q_h(\omega_1), \nabla w_h)_{\mathcal{T}} - (w_0 - w_b, [\nabla(\omega_1) - Q_h(\nabla(\omega_1))] \cdot n)_{\partial T}).\]

(3.42)

By the trace estimate (3.1), (3.19), and the boundedness of $\omega_1$, it follows that

\[|\sum_{T \in T_{D_2}} (w_0 - w_b, [\nabla(\omega_1) - Q_h(\nabla(\omega_1))] \cdot n)_{\partial T}| \leq C \sum_{T \in T_{D_2}} h^{-1/2}_T ||w_0 - w_b||_{0,\partial T} h^{l+1}_T ||\omega_1||_{l+2,T}
\leq Ch^{l+1} S_{D_2}(w_0, w_h)^{1/2}||\omega||_{l+2,D_1}.\]

(3.43)

Meanwhile, since $Q_h(\omega_1) \in \tilde{V}_{D_2,h} \subset \tilde{V}_{D_3,h}$, by (3.29), (2.8), and $\omega_2|_{\partial D_2} = 1$, we have

\[a(Q_h(\omega_1), w_h) = a(Q_h(\omega_1), S(\omega_2) - u_h)
= (-\Delta u + \Delta(\omega_2 u), Q_0(\omega_1)) = 0.\]

Therefore, by the estimate above, $Q_h Q_0|_c = Q_0|_c$ for $c \in \mathcal{E}_h$, the trace estimate, and (2.3), we have

\[|\nabla w(Q_h(\omega_1), \nabla w_h)| = |\sum_{T \in T_{D_2}} h^{-1}_T (w_0 - w_b, Q_h(\omega_1) - Q_0(\omega_1))_{\partial T}|
\leq C \left( \sum_{T \in T_{D_2}} h^{-1}_T ||w_0 - w_b||^2_{0,\partial T} \right)^{1/2} \left( \sum_{T \in T_{D_2}} h^{-1}_T ||Q_h(\omega_1) - Q_0(\omega_1)||^2_{0,\partial T} \right)^{1/2}
\leq C h^{l+1} S_{D_2}(w_0, w_h)^{1/2}||\omega||_{l+2,D_1}.\]

(3.44)
By $\omega_2|_{\bar{D}_2} = 1$ and Proposition 2.2, it holds that
\[
s_{\bar{D}_2}(w_h, w_h)^{1/2} \leq \|\omega_2 u - S(\omega_2 u)\|_{h, \bar{D}_2} + s_{\bar{D}_2}(u - u_h, u - u_h)^{1/2}
\]
\begin{equation}
(3.45)
\end{equation}
Due to (2.3) and (2.4), the first term in (4.1) represents the best approximation error.
\[
\|u - u_0\|_{l, D_0} \leq C(h^{t+k+1}\|u\|_{l+1, D_2} + h^{t+k+1}s_{D_2}(u_h, u_h)^{1/2} + \|u - u_0\|_{l-1, D_2}),
\]
which completes the proof. \[ \square \]

4. Interior Energy Estimates. In this section, we derive our main results for
the WG finite element solution in interior regions of the computational domain. In
particular, for $D_0 \subset \subset \Omega$, we obtain estimates on the local error in the energy norm
$\|u - u_h\|_{h, D_0}$.
We fix the notation by carrying out the analysis on the concentric balls $D_0 \subset \subset D_1 \subset \subset D_2 \subset \subset D_3 \subset \subset D \subset \subset \Omega$. The estimate on general interior regions will follow from
a covering argument. Note that based on the definition in (3.28), we have
\[
\|u - u_h\|_{h, D_0} = \|\omega_1 u - u_h\|_{h, D_0} \leq \|\omega_1 u - Q_h(\omega_1 u)\|_{h, \bar{D}_2} + \|Q_h(\omega_1 u) - u_h\|_{h, D_0}.
\]
Due to (2.3) and (2.4), the first term in (4.1) represents the best approximation error
to $\omega_1 u$ in an interior region $\bar{D}_2$. Hence, we shall focus on the analysis for the second
term $\|Q_h(\omega_1 u) - u_h\|_{h, D_0}$.
Note that for $v = \{v_0, v_b\}, w = \{w_0, w_b\} \in V_h$ and $T \in T_h$, we have $(v, w)_T = (v_0, w_0)_T$ and $v|_T = v_0|_T$. Then, we have the following lemma.

**Lemma 4.1.** Recall the local discrete space in (2.1). Using the bilinear form $a(\cdot, \cdot)$
in (2.7), we define the projection $P_{D_2}$ onto $\tilde{V}_{D_2,h}$, such that for $v \in V_h$,
\[
a(\omega_1 v - P_{D_2}(\omega_1 v), w) = 0, \quad \forall w \in \tilde{V}_{D_2,h}.
\]
Then, for $h$ sufficiently small, we have
\begin{equation}
(4.2)
\end{equation}
\[
\|\omega_1 v - P_{D_2}(\omega_1 v)\|_{h, \bar{D}_2} + \|\omega_1 v - Q_h(\omega_1 v)\|_{\bar{D}_2} \leq Ch\|v\|_{\bar{D}_2},
\]
\[
(4.3)
\end{equation}

**Proof.** Based on the definition (2.9), $\text{supp} \omega_1 \subseteq \bar{D}_2$, and (4.2), for any $w \in \tilde{V}_{D_2,h}$,
we have
\[
\|\omega_1 v - P_{D_2}(\omega_1 v)\|^2_{\bar{D}_2} = a(\omega_1 v - P_{D_2}(\omega_1 v), \omega_1 v - P_{D_2}(\omega_1 v))
\end{equation}
\[
(4.4)
\end{equation}
\[
= a(\omega_1 v - P_{D_2}(\omega_1 v), \omega_1 v - w)
\]
\[
\leq \|\omega_1 v - P_{D_2}(\omega_1 v)\|_{\bar{D}_2}\|\omega_1 v - w\|_{\bar{D}_2}.
\]
Theorem 3.28, there exists $C > 0$ independent of $h$, such that $|D^l \omega_1| < C$ for $0 \leq l \leq k + 1$. Therefore, for $T \in T_{D_2}$, by the definitions in (2.10) and (2.9),
and by (3.5), we have
\[
\|\omega_1 v - P_{D_2}(\omega_1 v)\|_{h, T} \leq \|\nabla (\omega_1 v - P_{D_2}(\omega_1 v))\|_{0, T} + \|\omega_1 v - P_{D_2}(\omega_1 v)\|_{T}
\]
\[
\leq \|\nabla (\omega_1 v - Q_h(\omega_1 v))\|_{0, T} + \|\nabla (Q_h(\omega_1 v) - P_{D_2}(\omega_1 v))\|_{0, T} + \|\omega_1 v - P_{D_2}(\omega_1 v)\|_{T}
\]
\[
\leq C h_T \|\omega_1 v\|_{1, T} + \|\nabla (Q_h(\omega_1 v) - P_{D_2}(\omega_1 v))\|_{0, T} + \|\omega_1 v - P_{D_2}(\omega_1 v)\|_{T}.
\]
Taking the summation of the above inequality over all \(T \in T_{D_2}\), by \(\text{supp}\omega_1 \subseteq D_1\), (3.2), and (4.4), we have
\[
\|\omega_1 v - P_{D_2}(\omega_1 v)\|_{h, D_2}
\]
\[
\leq C \left( \sum_{T \in T_{D_2}} \|v_0\|_{1, T}^{1/2} + \|Q_h(\omega_1 v) - P_{D_2}(\omega_1 v)\|_{\bar{D}_2} + \|\omega_1 v - P_{D_2}(\omega_1 v)\|_{D_2} \right)
\]
\[
\leq C \left( \sum_{T \in T_{D_2}} \|v_0\|_{1, T}^{1/2} + \|Q_h(\omega_1 v) - \omega_1 v\|_{\bar{D}_2} + \|\omega_1 v - P_{D_2}(\omega_1 v)\|_{D_2} \right)
\]
\[
(4.5) \leq C \left( \sum_{T \in T_{D_2}} \|v_0\|_{1, T}^{1/2} + \|Q_h(\omega_1 v) - \omega_1 v\|_{\bar{D}_2} \right).
\]
Let \(\psi = Q_h(\omega_1 v) - \omega_1 v\). Recall \(\nabla_w \psi|_T \in [P_{k-1}(T)]^d\) and
\[
\|\nabla_w \psi\|_{0,T} = \sum_{T \in T_{D_2}} \langle \nabla_w \psi, \nabla_w \psi \rangle_T + h_T^{-1} \langle \psi - \psi_b, \psi - \psi_b \rangle_T.
\]
Then, by (2.2), the trace estimate (3.1) and the inverse inequality, we have
\[
\|\nabla_w \psi\|_{0,T} = \sup_{0 \not\equiv q \in [P_{k-1}(T)]^d} \frac{\langle \nabla_w \psi, q \rangle_T}{\|q\|_{0,T}} = \sup_{0 \not\equiv q \in [P_{k-1}(T)]^d} \frac{\langle \nabla \psi, q \rangle_T + \langle \psi - \psi_b, q \cdot \mathbf{n} \rangle_{\partial T}}{\|q\|_{0,T}}
\]
\[
\leq \|\nabla \psi\|_{0,T} + C \|\psi - \psi_b\|_{0, \partial T} (h_T^{-1/2} \|q\|_{0,T} + h_T^{1/2} \|\nabla q\|_{0,T})
\]
\[
(4.7) \leq \|\nabla \psi\|_{0,T} + C h_T^{-1/2} \|\psi - \psi_b\|_{0, \partial T}.
\]
Then, by (3.4), (3.6), the boundedness of \(\omega_1\), the trace estimate, (3.5), and the inverse inequality, we further have
\[
\|\psi_b - \psi_b\|_{0, \partial T} = \|Q_b(\omega_1 v_b) - \omega_1 v_b - (Q_0(\omega_1 v_0) - \omega_1 v_0)\|_{0, \partial T}
\]
\[
\leq \|\omega_1 v_0 - \omega_1 v_b - Q_b(\omega_1 v_0 - \omega_1 v_b)\|_{0, \partial T}
\]
\[
+ \|\omega_1 v_0 - Q_b(\omega_1 v_0)\|_{0, \partial T} + \|\omega_1 v_0 - Q_0(\omega_1 v_0)\|_{0, \partial T}
\]
\[
\leq C (h_T \|v_0 - v_b\|_{0, \partial T} + \|\omega_1 v_0 - Q_0(\omega_1 v_0)\|_{0, \partial T})
\]
\[
\leq C (h_T \|v_0 - v_b\|_{0, \partial T} + h_T^{-1/2} \|\omega_1 v_0 - Q_0(\omega_1 v_0)\|_{0, T}
\]
\[
+ h_T^{1/2} \|\omega_1 v_0 - Q_0(\omega_1 v_0)\|_{0, T}
\]
\[
\leq C h_T \|v_0 - v_b\|_{0, \partial T} + h_T^{3/2} \|v_0\|_{1,T}.
\]
Thus, by (4.5) – (4.7), (3.5), (3.2), and Proposition 2.1, we have
\[
\|\nabla (\omega_1 v - P_{D_2}(\omega_1 v))\|_{h, D_2} + \|\omega_1 v - Q_h(\omega_1 v)\|_{\bar{D}_2}
\]
\[
\leq C \left( \sum_{T \in T_{D_2}} h_T^2 \|v_0\|_{1,T}^{2} + \|\nabla \psi\|_{0,T}^2 + h_T \|v_0 - v_b\|_{0,\partial T} + h_T^2 \|v_0\|_{1,T}^{2} \right)^{1/2}
\]
\[
\leq C h \left( \sum_{T \in T_{D_2}} \|v_0\|_{1,T}^{2} + h_T^{-1} \|v_0 - v_b\|_{0,\partial T}^2 \right)^{1/2} \leq C h \|v\|_{\bar{D}_2}.
\]
which completes the proof. \[ \square \]

Next, we derive an equation regarding $Q_h(\omega_3 u) - u_h$ that will be useful to carry out further analysis.

**Lemma 4.2.** Define $z = Q_h(\omega_3 u) - u_h \in V_h$. For $T \in T_h$ and $v \in V_h$, we have

$$
(\nabla_w(\omega_1 z), \nabla_w v)_T = (\nabla_w z, \nabla_w (\omega_1 v))_T + (z_0 \nabla \omega_1, \nabla v)_T + (z_0, \nabla \cdot (v_0 \nabla \omega_1))_T
$$

\[ (4.8) \]

$$
-\langle z_0, v_0 \nabla \omega_1 \cdot n \rangle_{\partial T} + \zeta_T(z, v, \omega_1) + \ell_T(z, v, \omega_1),
$$

where

$$
\zeta_T(z, v, \omega_1) = (v_b - v_0, Q_h \nabla (\omega_1 z_0) \cdot n - \omega_1 \nabla \omega_1 \cdot n)_{\partial T},
$$

$$
\ell_T(z, v, \omega_1) = (z_b - z_0, \omega_1 \nabla w_v \cdot n - Q_h \nabla (\omega_1 v_0) \cdot n)_{\partial T}.
$$

**Proof.** It follows from the product rule and integration by parts that

$$
(\nabla (\omega_1 z_0), \nabla v)_T = (\nabla z_0, \nabla (\omega_1 v_0))_T + (z_0 \nabla \omega_1, \nabla v)_T + (z_0, \nabla \cdot (v_0 \nabla \omega_1))_T
$$

\[ (4.9) \]

$$
-\langle z_0, v_0 \nabla \omega_1 \cdot n \rangle_{\partial T}. 
$$

Using (2.2) and integration by parts, we have for $w \in W(T)$ and $v \in V_h$,

$$
(\nabla_w w, \nabla_w v)_T = -(w_0, \nabla \cdot \nabla_w v)_T + \langle w_b, \nabla_w v \cdot n \rangle_{\partial T}
$$

\[ = (\nabla w_0, \nabla_w v)_T + \langle w_b - w_0, \nabla_w v \cdot n \rangle_{\partial T} 
\]

$$
= -(v_0, \nabla \cdot Q_h \nabla w_0)_T + \langle v_b, Q_h \nabla w_0 \cdot n \rangle_{\partial T} + \langle w_b - w_0, \nabla_w v \cdot n \rangle_{\partial T}
$$

\[ (4.10) \]

Letting $w = \omega_1 z$ in (4.10), we get

$$
(\nabla_w (\omega_1 z), \nabla_w v)_T = (\nabla (\omega_1 z_0), \nabla v)_T + \langle v_b - v_0, Q_h \nabla (\omega_1 z_0) \cdot n \rangle_{\partial T}
$$

\[ + \langle z_b - z_0, \omega_1 \nabla w_v \cdot n \rangle_{\partial T}. \]

(4.11)

Similarly, we have

$$
(\nabla_w z, \nabla_w (\omega_1 v))_T = (\nabla z_0, \nabla (\omega_1 v_0))_T + \langle z_b - z_0, Q_h \nabla (\omega_1 v_0) \cdot n \rangle_{\partial T}
$$

\[ + \langle v_b - v_0, \omega_1 \nabla w_z \cdot n \rangle_{\partial T}. \]

(4.12)

Therefore, by (4.11), (4.9), and (4.12), we obtain

$$
(\nabla_w (\omega_1 z), \nabla_w v)_T = (\nabla (\omega_1 z_0), \nabla v)_T + \langle v_b - v_0, Q_h \nabla (\omega_1 z_0) \cdot n \rangle_{\partial T}
$$

\[ + \langle z_b - z_0, \omega_1 \nabla w_v \cdot n \rangle_{\partial T} 
\]

$$
= (\nabla z_0, \nabla (\omega_1 v_0))_T + \langle v_b - v_0, Q_h \nabla (\omega_1 z_0) \cdot n \rangle_{\partial T}
$$

$$
- \langle z_b - z_0, \omega_1 \nabla w_v \cdot n \rangle_{\partial T}
$$

$$
- \langle z_0, v_0 \nabla \omega_1 \cdot n \rangle_{\partial T} + \langle v_b - v_0, Q_h \nabla (\omega_1 z_0) \cdot n \rangle_{\partial T}
$$

\[ + \langle v_b - v_0, \omega_1 \nabla w_z \cdot n \rangle_{\partial T} 
\]

$$
= (\nabla_w (\omega_1 z), \nabla_w v)_T + \langle z_b - z_0, Q_h \nabla (\omega_1 v_0) \cdot n \rangle_{\partial T}
$$

$$
- \langle z_b - z_0, \omega_1 \nabla w_z \cdot n \rangle_{\partial T}
$$

$$
- \langle v_b - v_0, \omega_1 \nabla w_z \cdot n \rangle_{\partial T}
$$

\[ + \langle z_b - z_0, \omega_1 \nabla w_z \cdot n \rangle_{\partial T} 
\]

$$
- \langle z_0, v_0 \nabla \omega_1 \cdot n \rangle_{\partial T} + \zeta_T(z, v, \omega_1) + \ell_T(z, v, \omega_1). 
$$
This completes the proof. \( \Box \)

In addition, we obtain an estimate on the difference between the weak gradient and the usual gradient.

**Lemma 4.3.** For \( T \in \mathcal{T}_h \) and \( v \in V_h \), we have

\[
\| \nabla_w v - \nabla v_0 \|_T \leq C h_T^{-1/2} \| v_0 - v_b \|_{\partial T}.
\]

**Proof.** It follows from the definition of weak gradient (2.2) that for any \( q \in [P_{k-1}(T)]^d \),

\[
(\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + (v_b, q \cdot n)_{\partial T} = (\nabla v_0, q)_T + (v_b - v_0, q \cdot n)_{\partial T}.
\]

Letting \( q = \nabla_w v - \nabla v_0 \) in the above equation and using the trace inequality (3.1) and the inverse inequality, we have

\[
\| \nabla_w v - \nabla v_0 \|_T^2 \leq \langle v_b - v_0, (\nabla_w v - \nabla v_0) \cdot n \rangle_{\partial T} \leq C h_T^{-1/2} \| v_0 - v_b \|_{\partial T} \| \nabla_w v - \nabla v_0 \|_T,
\]

which proves the lemma. \( \Box \)

With the estimates developed above, we proceed to analyze the local error \( \| u - u_h \|_{h,D_0} \) in the interior of the domain. Recall \( z = Q_h(\omega_3 u) - u_h \) in Lemma 4.2. Then, we further have the following estimate on \( z \).

**Lemma 4.4.** Recall the bilinear form \( s(\cdot, \cdot) \) in (2.7) and its restriction \( s_{D_2}(\cdot, \cdot) \) on \( D_2 \). Suppose \( u \in H^{\mu+1}(D_2) \) for some \( \mu > 1/2 \). Let \( t = \min(\mu, k) \) and \( h \) be sufficiently small. Then, for any \( v \in V_{D_2,h} \), we have

\[
a(\omega_1 z, v) \leq C(h^t \| u \|_{t+1,D_2} + \| u - u_0 \|_{0,D_2} + hs_{D_2}(u_h, u_h)^{1/2}) \| v \|_{D_2},
\]

where \( u_h = \{u_0, u_k\} \in V_h \) is the WG approximation in (2.8).

The proof of the lemma is long and can be found in Appendix.

Next, we are ready to present the interior energy estimate for the WG finite element approximation.

**Theorem 4.5.** Let \( u \) and \( u_h = \{u_0, u_k\} \in V_h \) be the solution of equation (1.1) and its WG approximation (2.8), respectively. Let \( D_0 \subset \subset D \subset \subset \Omega \) be interior regions. Suppose the solution of equation (1.1) \( u \in H^{\mu+1}(D) \) for some \( \mu > 1/2 \). Let \( t = \min(\mu, k) \). Then, for \( 0 \leq l \leq k - 1 \) and for \( h \) sufficiently small, we have

\[
\| u - u_h \|_{h,D_0} \leq C(h^t \| u \|_{t+1,D} + \sum_{j=0}^{l} h^{l-j} \| u - u_0 \|_{-j,D} + h^{l+1} s_D(u - u_h, u - u_h)^{1/2}).
\]

**Proof.** We show the proof for concentric balls \( D_0 \subset \subset D_1 \subset \subset D_2 \subset \subset D_3 \subset \subset D \). The case for a general interior region follows from a covering argument using these balls.

By the definition (3.28), we have \( Q_h(\omega_1 u)|_{D_0} = Q_h(\omega_3 u)|_{D_0} \). Following (4.1) and using (2.12) for \( \| \omega_1 u - Q_h(\omega_1 u) \|_{h, D_2} \), by the boundedness of \( \omega_1 \), we obtain

\[
\| u - u_h \|_{h,D_0} \leq \| \omega_1 u - Q_h(\omega_1 u) \|_{h, D_2} + \| Q_h(\omega_3 u) - u_h \|_{h,D_0}
\leq Ch^t \| u \|_{t+1,D_2} + \| Q_h(\omega_3 u) - u_h \|_{h,D_0}.
\]

(4.16)
Recall $z = Q_h(\omega_3 u) - u_h$ in Lemma 4.4. It follows from (4.3), (3.2), (3.3), the inverse inequality, (A.11), and (A.12) that

\[
\|z\|_{h,D_0} = \|\omega_3 z\|_{h,D_0} \\
\leq \|\omega_3 z - P_{D_2}(\omega_3 z)\|_{h,D_0} + \|P_{D_2}(\omega_3 z)\|_{h,D_0} \\
\leq Ch\|z\|_{D_2} + \|P_{D_2}(\omega_3 z)\|_{h,D_0} \\
\leq C\|z\|_{0,D_2} + h s_{D_2}(z,z)^{1/2} + \|P_{D_2}(\omega_3 z)\|_{D_2} \\
(4.17) \\
\leq C(h^{t+1}\|u\|_{l+1,D_2} + \|u - u_0\|_{0,D_2} + h s_{D_2}(u_h, u_h)^{1/2}) + \|P_{D_2}(\omega_3 z)\|_{D_2}.
\]

Using (4.15), we have

\[
\|P_{D_2}(\omega_3 z)\|_{D_2} = \sup_{0 \neq v \in V_{D_2,h}} \frac{a(P_{D_2}(\omega_3 z), v)}{\|v\|_{D_2}} = \sup_{0 \neq v \in V_{D_2,h}} \frac{a(\omega_3 z, v)}{\|v\|_{D_2}} \\
(4.18) \\
\leq C(h^t\|u\|_{l+1,D_2} + \|u - u_0\|_{0,D_2} + h s_{D_2}(u_h, u_h)^{1/2}).
\]

Thus, by (4.16) – (4.18), we derive

\[
(4.19) \\
\|u - u_h\|_{h,D_0} \leq C(h^t\|u\|_{l+1,D_2} + \|u - u_0\|_{0,D_2} + h s_{D_2}(u_h, u_h)^{1/2}).
\]

For any $l \geq 0$, we introduce a set of concentric balls $D_2 \subset D_3 \subset \cdots \subset D_{l+2} = D$. Note that $s_{D_2}(u_h, u_h)^{1/2} = s_{D_2}(u - u_h, u - u_h)^{1/2}$ is part of the local energy error in the interior region $D_2$. Then, the estimate (4.19) leads to

\[
(4.20) \\
h s_{D_2}(u - u_h, u - u_h)^{1/2} \leq C(h^t\|u\|_{l+1,D_2} + \|u - u_0\|_{0,D_2} + h s_{D_2}(u_h, u_h)^{1/2}).
\]

By (4.20) and the interior negative norm estimate in Theorem 3.7, it holds that

\[
\|u - u_0\|_{0,D_2} + h s_{D_2}(u_h, u_h)^{1/2} \\
\leq C(h^{l+1}\|u\|_{l+1,D_2} + h s_{D_2}(u_h, u_h)^{1/2}) \\
\leq C(h^{l+1}\|u\|_{l+1,D_2} + h\|u - u_0\|_{0,D_3} + \|u - u_0\|_{-1,D_3} + h^2 s_{D_2}(u_h, u_h)^{1/2}).
\]

Repeating this process $l$ times, we have

\[
\|u - u_0\|_{0,D_2} + h s_{D_2}(u_h, u_h)^{1/2} \leq C(h^{l+1}\|u\|_{l+1,D} \\
+ \sum_{j=0}^{l} h^{l-j}\|u - u_0\|_{-j,D} + h^{l+1}s_{D}(u_h, u_h)^{1/2}).
\]

Therefore, combining this estimate with (4.19), we derive

\[
\|u - u_h\|_{h,D_0} \leq C(h^t\|u\|_{l+1,D} + \sum_{j=0}^{l} h^{l-j}\|u - u_0\|_{-j,D} + h^{l+1}s_{D}(u_h, u_h)^{1/2}) \\
= C(h^t\|u\|_{l+1,D} + \sum_{j=0}^{l} h^{l-j}\|u - u_0\|_{-j,D} + h^{l+1}s_{D}(u - u_h, u - u_h)^{1/2}),
\]

which completes the proof. \(\qed\)

According to Theorem 4.5, the interior energy error is bounded by the best local approximation in the WG finite element space $h^t\|u\|_{l+1,D}$, the interior negative
norms of the error $\sum_{j=0}^{l} h^{l-j} \|u - u_0\|_{-j,D}$, and the error on the element boundaries $h^{l+1} s_D(u - u_h, u - u_h)^{1/2}$. The last two components achieve the highest order of convergence when $l = k - 1$. We here discuss the impact of the last two components on the interior energy convergence of the WG approximation.

**Remark 4.6.** Taking $l = k - 1$ in Theorem 4.5, we observe that the error on the element boundaries $h^{k} s_D(u - u_h, u - u_h)^{1/2}$ is of high order. Therefore, the interior energy error is in fact bounded by the best local approximation $h^l \|u\|_{l+1,D}$ and the interior negative norms of the error $\sum_{j=0}^{k-1} h^{k-1-j} \|u - u_0\|_{-j,D}$. Using the global negative norm estimate (3.9), for each interior negative norm, it holds that

$$h^{k-1-j} \|u - u_0\|_{-j,D} \leq \begin{cases} Ch^{l+k} \|u\|_{l+1,\Omega} & \text{if } 0 \leq j \leq p - 1, \\ Ch^{l+p-1-j} \|u\|_{l+1,\Omega} & \text{if } j > p - 1, \end{cases}$$

where as defined in Theorem 3.4, $l' = \min(\mu, k)$ with $\mu$ as the global regularity parameter for the solution $u$ (namely, $u \in H^{l+1}(\Omega)$) and $p = \min(a, k)$. Recall the regularity index $\eta > 1/2$ (see (3.8)) for the dual problem (3.7). In the case $\eta > k$, we can take $p = k$ such that the negative norms of the error are of higher order. Therefore, the interior energy norm of the error is bounded by the best local approximation error. In the case $\eta < k$, the negative norms of the error can be the dominant terms in the interior energy error estimate. For example, letting $j = k - 1$ in the estimate above, we have $\|u - u_0\|_{-k+1,D} \leq Ch^{l+p} \|u\|_{l+1,\Omega}$, which may not be better than the best local approximation error. This implies that the WG solution in the interior of the domain can be polluted by the singularities (associated to both the original and the dual problems) in other parts of the domain. This phenomenon resembles the local behavior of the usual finite element method, which has been well documented in [19, 23, 26].

5. **Numerical Results.** In this section, we report numerical results to illustrate the theoretical findings in the previous sections. We implement the WG methods associated with the quasi-uniform triangulations $T_h$ on two domains, the L-shaped domain and the square, to solve the Poisson equation.

![Computational domains: the L-shaped domain $\Omega = (-1,1)^2 \setminus [(0,1) \times (-1,0)]$ and the shaded interior region $D = (-1/4,3/4) \times (1/4,3/4)$ (left); the square domain $\Omega = (0,1)^2$ and the shaded interior region $D = (1/4,3/4) \times (1/4,3/4)$ (right).](image)

**5.1. Test 1.** The first test is on the L-shaped domain $\Omega = (-1,1)^2 \setminus [(0,1) \times (-1,0)]$ (Figure 5.1). Note that the regularity index $\eta$ of the dual problem (3.7) (see also (3.8)) satisfies $\eta = \pi/\chi = 2/3$ on $\Omega$, where $\chi = 3\pi/2$ is the largest opening angle associated to the vertex of the domain. We choose the exact solution that possesses
the typical singularity near the reentrant corner at the origin
\[ u = r^{2/3} \sin \left( \frac{2\theta}{3} \right) - \frac{r^2}{4}, \quad r = \sqrt{x^2 + y^2}. \]

Therefore, \( u \in H^{5/3-\epsilon}(\Omega) \) for \( \epsilon > 0 \) arbitrarily small. Based on Proposition 2.2 and Theorem 3.4, for any polynomial degree \( k \geq 1 \), the global energy convergence rate of the WG solution is \( O(h^{2/3-\epsilon}) \), and the convergence rate in the global negative norm of order \( l \leq 0 \) is \( O(h^{4/3-\epsilon}) \). In the interior region \( D = (-1/4, 3/4) \times (1/4, 3/4) \), the solution \( u \) is smooth. Therefore, the estimate in Theorem 4.5 (see also Remark 4.6) shows that the local energy convergence rate on \( D \) for the WG method should be \( O(h^{\min(k,4/3-\epsilon)}) \) instead of \( h^k \).

**Table 5.1**

<table>
<thead>
<tr>
<th>( 1/h )</th>
<th>local error on ( D )</th>
<th>global error on ( \Omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>energy error</td>
<td>rate</td>
</tr>
<tr>
<td>( k = 1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2.22E-02</td>
<td>7.09E-02</td>
</tr>
<tr>
<td>16</td>
<td>1.03E-02</td>
<td>1.10</td>
</tr>
<tr>
<td>32</td>
<td>4.84E-03</td>
<td>1.09</td>
</tr>
<tr>
<td>64</td>
<td>2.29E-03</td>
<td>1.08</td>
</tr>
<tr>
<td>128</td>
<td>1.10E-03</td>
<td>1.06</td>
</tr>
<tr>
<td>256</td>
<td>5.34E-04</td>
<td>1.04</td>
</tr>
<tr>
<td>( k = 2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4.01E-03</td>
<td>3.52E-02</td>
</tr>
<tr>
<td>16</td>
<td>1.52E-03</td>
<td>1.40</td>
</tr>
<tr>
<td>32</td>
<td>5.94E-04</td>
<td>1.36</td>
</tr>
<tr>
<td>64</td>
<td>2.34E-04</td>
<td>1.34</td>
</tr>
<tr>
<td>128</td>
<td>9.28E-05</td>
<td>1.34</td>
</tr>
<tr>
<td>256</td>
<td>3.68E-05</td>
<td>1.33</td>
</tr>
<tr>
<td>( k = 3 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.74E-03</td>
<td>2.43E-02</td>
</tr>
<tr>
<td>16</td>
<td>6.32E-04</td>
<td>1.46</td>
</tr>
<tr>
<td>32</td>
<td>2.48E-04</td>
<td>1.35</td>
</tr>
<tr>
<td>64</td>
<td>9.86E-05</td>
<td>1.33</td>
</tr>
<tr>
<td>128</td>
<td>3.91E-05</td>
<td>1.33</td>
</tr>
<tr>
<td>256</td>
<td>1.55E-05</td>
<td>1.33</td>
</tr>
</tbody>
</table>

In Table 5.1 we display the convergence rates of the WG method with different polynomial degrees \( (k = 1, 2, 3) \). These results include global convergence rates and the local convergence rates in the interior region \( D \). It is clear from this table that the global energy norm convergence is always comparable to \( O(h^{2/3}) \) and the global \( L^2 \) norm convergence behaves like \( O(h^{4/3}) \). In the interior region, for \( k = 1 \), the energy norm convergence is of first order \( O(h) \), which is optimal for linear approximations. This is because, as mentioned above, the negative norms of the error in this case are of higher order \( O(h^{4/3-\epsilon}) \) and do not affect the local energy error.

For \( k = 2 \) and \( k = 3 \), based on our theory, the interior energy convergence rate shall be dominated by the convergence rate \( O(h^{4/3-\epsilon}) \) in negative norms. This can
be observed in Table 5.1. Hence, for \( k = 1, 2, 3 \), the numerical results are consistent with the theoretical predictions in Theorem 4.5.

### 5.2. Test 2

We also test the convergence of the WG methods with \( k = 1, 2 \) on the square domain \( \Omega = (0, 1)^2 \). In this case, the regularity index \( \eta \) for the dual problem (3.7) is given by \( \eta = \pi/\chi = 2 \), since the largest opening angle is \( \pi/2 \). We choose the exact solution as

\[
  u = x(1 - x)y(1 - y)r^{-3/2}.
\]

Therefore, \( u \in H^{3/2-\epsilon}(\Omega) \) for \( \epsilon > 0 \) arbitrarily small. Note that the singularity in \( u \) is around the origin. Based on Proposition 2.2, for the polynomial degree \( k = 1, 2 \), the global energy convergence rate of the WG solution is \( O(h^{1/2-\epsilon}) \). Meanwhile, because the dual problem is smoother (regularity index \( \eta = 2 \)), according to Theorem 3.4, when \( k = 1, 2 \), the convergence rate in the global \( L^2 \) norm shall be \( O(h^{1/2+k-\epsilon}) \).

In the interior region \( D = (-1/4, 3/4) \times (1/4, 3/4) \), the solution \( u \) is smooth. The estimate in Theorem 4.5 shows that the local energy convergence rate on \( D \) for the WG method should be comparable to the best local approximation rate \( O(h^k) \) \( (k = 1, 2) \).

### Table 5.2

<table>
<thead>
<tr>
<th>( 1/h )</th>
<th>local error on ( D )</th>
<th>global error on ( \Omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>energy error rate</td>
<td>energy error rate</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k = 1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>7.8595E-02 0.95</td>
<td>1.1146E+00 0.47</td>
</tr>
<tr>
<td>8</td>
<td>4.0821E-02 0.96</td>
<td>8.0396E-01 0.48</td>
</tr>
<tr>
<td>16</td>
<td>2.0975E-02 0.98</td>
<td>5.7520E-01 0.49</td>
</tr>
<tr>
<td>32</td>
<td>1.0636E-02 0.99</td>
<td>4.0958E-01 0.49</td>
</tr>
<tr>
<td>64</td>
<td>5.3518E-03 0.99</td>
<td>2.9081E-01 0.49</td>
</tr>
<tr>
<td>128</td>
<td>2.6831E-03 0.99</td>
<td>2.0612E-01 0.50</td>
</tr>
<tr>
<td>( k = 2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2.0011E-02 1.96</td>
<td>9.0488E-01 0.50</td>
</tr>
<tr>
<td>8</td>
<td>5.1444E-03 1.98</td>
<td>6.4193E-01 0.50</td>
</tr>
<tr>
<td>16</td>
<td>1.3038E-03 1.98</td>
<td>4.5494E-01 0.50</td>
</tr>
<tr>
<td>32</td>
<td>3.2732E-04 1.99</td>
<td>3.2211E-01 0.50</td>
</tr>
<tr>
<td>64</td>
<td>8.1943E-05 2.00</td>
<td>2.2793E-01 0.50</td>
</tr>
<tr>
<td>128</td>
<td>2.0496E-05 2.00</td>
<td>1.6123E-01 0.50</td>
</tr>
</tbody>
</table>

We list the numerical results of this test in Table 5.2. It is clear from the table that the interior energy convergence rate is always optimal \( O(h^k) \) \((k = 1, 2)\); and the convergence rates in global energy norm and in the global \( L^2 \) norm are \( O(h^{1/2-\epsilon}) \) and \( O(h^{3/2-\epsilon}) \), respectively. This is in agreement with our theory developed in previous sections.

### Appendix A

The proof of Lemma 4.4.
Proof. It follows from (4.8), (2.7), and supp($\omega_1$) $\subseteq D_1$, that

$$a(\omega_1 z, v) = a(z, \omega_1 v) + \sum_{T \in T_{D_2}} (z_0 \nabla \omega_1, \nabla v_0)_T + \sum_{T \in T_{D_2}} (z_0, \nabla \cdot (v_0 \nabla \omega_1))_T$$

$$+ \sum_{T \in T_{D_2}} (\ell_T(z, v, \omega_1) - (z_0, v_0 \nabla \omega_1 \cdot n)_{\partial T}) + \sum_{T \in T_{D_2}} \zeta_T(z, v, \omega_1)$$

(A.1)

$$= I_1 + I_2 + I_3 + I_4 + I_5.$$ 

Below, we obtain estimates for each $I_i, 1 \leq i \leq 5$.

Recall supp($\omega_1 v$) $\subseteq D_1$. Then, using (2.11), (2.13), (2.14), (4.3), (3.3), and the inverse inequality, we have

$$|I_1| = |a(z, \omega_1 v)| = |a(Q_h u - u_h, \omega_1 v - Q_h(\omega_1 v)) + a(Q_h u - u_h, Q_h(\omega_1 v))|$$

$$\leq \|Q_h u - u_h\|_{D_2} \|\omega_1 v - Q_h(\omega_1 v)\|_{D_2} + Ch\|u\|_{t+1, D_2} \|v\|_{D_1}$$

$$\leq Ch\|z\|_{D_2} \|v\|_{D_2} + Ch\|u\|_{t+1, D_2} \|v\|_{D_2}$$

(A.2)

$$\leq C(h^{\ell} \|u\|_{t+1, D_2} + \|z_0\|_{0, D_2} + hsD_2(z, z^{1/2}) \|v\|_{D_2}.$$ 

Using the boundedness of $\omega_1$, (3.2), and the Cauchy-Schwarz inequality, we have

$$|I_2| = \left| \sum_{T \in T_{D_2}} (z_0 \nabla \omega_1, \nabla v_0)_T \right| \leq C\|z\|_{0, D_2} \|v\|_{D_2}$$

(A.3)

and

$$|I_3| = \left| \sum_{T \in T_{D_2}} (z_0, \nabla \cdot (v_0 \nabla \omega_1))_T \right| \leq C\|z\|_{0, D_2} \|v\|_{D_2}.$$ 

(A.4)

Now, we estimate $I_4$ and $I_5$. Note that $v_b = 0$ on $\partial D_2$. Therefore,

$$\sum_{T \in T_{D_2}} (z_b, v_b \nabla \omega_1 \cdot n)_{\partial T} = 0.$$ 

Then, we have

$$I_4 = \sum_{T \in T_{D_2}} ((z_0 - z_b, Q_h \nabla (\omega_1 v_0) \cdot n - \omega_1 \nabla w \cdot n)_{\partial T} - (z_0, v_0 \nabla \omega_1 \cdot n)_{\partial T})$$

$$= \sum_{T \in T_{D_2}} ((z_0 - z_b, (Q_h(\omega_1 \nabla v_0) - \omega_1 \nabla w) \cdot n)_{\partial T}$$

$$+ (z_0 - z_b, (Q_h(v_0 \nabla \omega_1) - v_0 \nabla \omega_1) \cdot n)_{\partial T}$$

$$+ (z_0 - z_b, (v_0 - v_b) \nabla \omega_1 \cdot n)_{\partial T} - (v_0 - v_b, z_0 \nabla \omega_1 \cdot n)_{\partial T})$$

(A.5)

$$= A_1 + A_2 + A_3 - A_4.$$ 

By the Cauchy-Schwarz inequality, the trace inequality (3.1), the boundedness of $\omega_1$, ...
(2.5), the inverse inequality, and (3.2), we have
\[
|A_2 + A_3| \leq \sum_{T \in T_{D_2}} \|z_0 - z_b\|_{0,\partial T} \|((Q_h(v_0 \nabla \omega_1)) - v_0 \nabla \omega_1) \cdot n\|_{0,\partial T}
+ \|((v_0 - v_b) \nabla \omega_1 \cdot n\|_{0,\partial T})
\leq C \left( \sum_{T \in T_{D_2}} \|z_0 - z_b\|_{0,\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_{D_2}} \|((Q_h(v_0 \nabla \omega_1)) - v_0 \nabla \omega_1) \cdot n\|_{0,\partial T}^2 \right)
+ h |Q_h(v_0 \nabla \omega_1) - v_0 \nabla \omega_1|^2_{0, \partial T} + \|v_0 - v_b\|_{0,\partial T}^2)^{1/2}
\leq C \left( \sum_{T \in T_{D_2}} \|z_0 - z_b\|_{0,\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_{D_2}} \|v_0 \nabla \omega_1\|_{0, \partial T}^2 + \|v_0 - v_b\|_{0,\partial T}^2 \right)
\leq C h \|s_{D_2}(z, z)\|_{1/2}^2 \|v\|_{D_2}.
\]
(A.6)

Similarly, for $A_4$, we have
\[
|A_4| \leq C \left( \sum_{T \in T_{D_2}} \|v_0 - v_b\|_{0,\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_{D_2}} \|h^{-1} \|z_0 \nabla \omega_1\|_{0, \partial T}^2 + h |z_0 \nabla \omega_1|^2_{0, \partial T} \right)
\leq C \left( \sum_{T \in T_{D_2}} \|v_0 - v_b\|_{0,\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_{D_2}} \|z_0\|^2_{0, \partial T} \right)^{1/2} \leq C \|z_0\|_{0, \partial D_2} \|v\|_{D_2}.
\]
(A.7)

Now, for $A_1$, we first have
\[
A_1 = \sum_{T \in T_{D_2}} \langle z_0 - z_b, (Q_h(\omega_1 \nabla v_0) - \omega_1 \nabla w v) \cdot n)_{\partial T}
= \sum_{T \in T_{D_2}} \langle z_0 - z_b, Q_h(\omega_1 \nabla v_0 - \omega_1 \nabla w v) \cdot n)_{\partial T}
+ \sum_{T \in T_{D_2}} \langle z_0 - z_b, (Q_h(\omega_1 \nabla w v) - \omega_1 \nabla w v) \cdot n)_{\partial T}
= A_{11} + A_{12}.
\]
(A.8)

For the second term $A_{12}$, using the Cauchy-Schwarz inequality, the trace inequality
(3.1), the boundedness of $\omega_1$, (2.5), the fact $\partial^\alpha(\nabla_w v) = 0$ on $T$ for $|\alpha| = k$, and the
inverse inequality, we obtain
\[
|A_{12}| \leq \sum_{T \in T_{D_2}} \|z_0 - z_b\|_{0,\partial T} \|((Q_h(\omega_1 \nabla w v) - \omega_1 \nabla w v) \cdot n\|_{0,\partial T}
\leq C \left( \sum_{T \in T_{D_2}} \|z_0 - z_b\|_{0,\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_{D_2}} \|Q_h(\omega_1 \nabla w v) - \omega_1 \nabla w v\|_{0, \partial T}^2 \right)
+ h |Q_h(\omega_1 \nabla w v) - \omega_1 \nabla w v|^2_{0, \partial T} \right)^{1/2}
\leq C \left( \sum_{T \in T_{D_2}} \|z_0 - z_b\|_{0,\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_{D_2}} h^{2k-1} \|\nabla w v\|^2_{0, \partial T} \right)^{1/2}
\leq C \left( \sum_{T \in T_{D_2}} \|z_0 - z_b\|_{0,\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_{D_2}} h \|\nabla w v\|^2_{0, \partial T} \right)^{1/2} \leq Ch s_{D_2}(z, z) \|v\|_{D_2}.
\]
(A.9)
The estimate on $A_{11}$ is more involved. Note that by (4.14), we have

$$A_{11} = \sum_{T \in \mathcal{T}_D} \langle z_0 - z_b, Q_h(\omega_1 \nabla v_0 - \omega_1 \nabla w) \cdot n \rangle_{\partial T}$$

$$= \sum_{T \in \mathcal{T}_D} (\nabla z_0 - \nabla w, Q_h(\omega_1 \nabla v_0 - \omega_1 \nabla w))_T$$

$$= \sum_{T \in \mathcal{T}_D} (Q_h(\omega_1 \nabla z_0 - \omega_1 \nabla w), \nabla v_0 - \nabla w)_T$$

$$= \sum_{T \in \mathcal{T}_D} \langle v_0 - v_b, Q_h(\omega_1 \nabla z_0 - \omega_1 \nabla w) \cdot n \rangle_{\partial T}.$$  

Therefore, considering $A_{11}$ and $I_5$ together, by the Cauchy-Schwarz inequality, the trace inequality (3.1), the boundedness of $\omega_1$, and (2.5), we obtain

$$|A_{11} + I_5| = |\sum_{T \in \mathcal{T}_D} \langle v_0 - v_b, Q_h(\omega_1 \nabla z_0 - \omega_1 \nabla w) \cdot n \rangle_{\partial T}$$

$$+ \sum_{T \in \mathcal{T}_D} (v_0 - v_b, (\omega_1 \nabla w - Q_h(\omega_1 \nabla z_0)) \cdot n)_{\partial T} + \sum_{T \in \mathcal{T}_D} (v_0 - v_b, Q_h(z_0 \nabla \omega_1) \cdot n)_{\partial T}|$$

$$= \sum_{T \in \mathcal{T}_D} \|v_0 - v_b\|_{0, \partial T} \|Q_h(\omega_1 \nabla z_0 - \omega_1 \nabla w)\|_{0, \partial T}$$

$$\leq C\left( \sum_{T \in \mathcal{T}_D} \|v_0 - v_b\|_{0, \partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_D} h^{-1} \|\omega_1 \nabla w - Q_h(\omega_1 \nabla z_0)\|_{0, T}^2 \right)^{1/2}$$

$$+ h\|\omega_1 \nabla w - Q_h(\omega_1 \nabla z_0)\|_{0, T}^2 \right)^{1/2}$$

$$+ \left( \sum_{T \in \mathcal{T}_D} \|v_0 - v_b\|_{0, \partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_D} h^{-1} \|Q_h(z_0 \nabla \omega_1)\|_{0, T}^2 + h \|Q_h(z_0 \nabla \omega_1)\|_{1, T}^2 \right)^{1/2}$$

$$\leq C\left( \sum_{T \in \mathcal{T}_D} \|v_0 - v_b\|_{0, \partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_D} h^{-1+2k} \|\omega_1 \nabla w\|_{k, T}^2 \right)^{1/2}$$

$$+ \left( \sum_{T \in \mathcal{T}_D} h^{-1} \|z_0 \nabla \omega_1\|_{0, T}^2 + h \|z_0 \nabla \omega_1\|_{0, T}^2 \right)^{1/2}.$$ 

Thus, by the inverse inequality, (3.3), and the fact $\partial^\alpha (\nabla w) = 0$ on $T$ for $|\alpha| = k$, we have

$$|A_{11} + I_5| \leq C\left( \sum_{T \in \mathcal{T}_D} \|v_0 - v_b\|_{0, \partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_D} h \|\nabla w\|_{0, T}^2 \right)^{1/2}$$

$$+ \left( \sum_{T \in \mathcal{T}_D} h^{-1} \|z_0\|_{0, T}^2 \right)^{1/2}$$

$$\leq C\|v\|_{\mathcal{D}_2} \left( \|z_0\|_{0, \mathcal{D}_2} + h \|z\|_{\mathcal{D}_2} \right)^{1/2}.  \tag{A.10}$$

It follows from the Cauchy-Schwarz inequality, the trace inequality, (2.3), (2.4),
and the inverse inequality that

\[
    s_{D_2}(z, z) = \sum_{T \in \mathcal{T}_{D_2}} s_T(Q_h u - u_h, z) = \sum_{T \in \mathcal{T}_{D_2}} s_T(Q_h u - u, z) - \sum_{T \in \mathcal{T}_{D_2}} s_T(u_h, z)
\]

\[
\leq C\left( \sum_{T \in \mathcal{T}_{D_2}} (h^{-2} \|u - Q_0 u_0\|_{0,T}^2 + |u - Q_0 u_0|^2_{1,T} + h^{-1}\|u - Q_0 u_0\|_{\partial \Omega,T}^2)^{1/2}
\right.
\]
\[
+ s_{D_2}(u_h, u_h)^{1/2}) s_{D_2}(z, z)^{1/2}
\]
\[
\leq C(h^t \|u\|_{t+1, D_2} + s_{D_2}(u_h, u_h)^{1/2}) s_{D_2}(z, z)^{1/2},
\]

which implies

\[
(A.11) \quad s_{D_2}(z, z)^{1/2} \leq C(h^t \|u\|_{t+1, D_2} + s_{D_2}(u_h, u_h)^{1/2}).
\]

In addition, the triangle inequality implies

\[
(A.12) \quad \|z_0\|_{0, D_2} \leq \|Q_h u_0 - u\|_{0, D_2} + \|u - u_0\|_{0, D_2} \leq C h^{t+1} \|u\|_{t+1, D_2} + \|u - u_0\|_{0, D_2}.
\]

Combining all the estimates (A.1) – (A.12), we have completed the proof for (4.15).

REFERENCES


