A SIMPLE FINITE ELEMENT METHOD FOR LINEAR HYPERBOLIC PROBLEMS

LIN MU * AND XIU YE†

Abstract. In this paper, we introduce a simple finite element method for solving first order hyperbolic equations with easy implementation and analysis. This new method, with a symmetric, positive definite system, is designed to use discontinuous approximations on finite element partitions consisting of arbitrary shape of polygons/polyhedra. Error estimate is established. Extensive numerical examples are tested that demonstrate the robustness and flexibility of the method.

Key words. finite element methods, hyperbolic equations

AMS subject classifications. Primary, 65N30, 65N15, 65N12, 74N20; Secondary, 35B45, 35J50, 35J35

1. Introduction. We consider the model problem that seeks an unknown function $u = u(x)$ satisfying

$$\nabla \cdot (\beta u) = f, \quad \text{in } \Omega,$$

$$u = g, \quad \text{on } \Gamma_-, \quad (1.1)$$

where $\beta$ is a flow field, $\Gamma_-$ is the inflow boundary defined as follows

$$\Gamma_- = \{x \in \partial \Omega, \; \beta(x) \cdot n(x) < 0\},$$

and $\Gamma_+ = \partial \Omega \setminus \Gamma_-$. We assume that the model problem (1.1) has a unique solution. This first order hyperbolic equation has many applications in science and engineering.

Linear hyperbolic partial differential equations may have discontinuous solutions when the boundary data is discontinuous. The nature of this type of problems makes it difficult to develop numerical methods for resolving discontinuities without introducing spurious oscillations. Numerical solutions of the first order hyperbolic equations have been investigated intensively and many different numerical schemes have been developed such as SUPG and continuous/discontinuous Galerkin methods [6, 9] and references therein.

The goal of this paper is to develop a simple and flexible finite element scheme to solve the hyperbolic equation (1.1). Our method can be derived by using discontinuous
2. Finite Element Formulation. For any given polygon \( D \subseteq \Omega \), we use the standard definition of Sobolev spaces \( H^s(D) \) with \( s \geq 0 \). The associated inner product, norm, and seminorms in \( H^s(D) \) are denoted by \( (\cdot, \cdot)_s,D, \| \cdot \|_s,D, \) and \( | \cdot |_{r,D}, 0 \leq r \leq s \), respectively. When \( s = 0 \), \( H^0(D) \) coincides with the space of square integrable functions \( L^2(D) \). In this case, the subscript \( s \) is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript \( D \) is also suppressed when \( D = \Omega \).

Let \( \mathcal{T}_h \) be a partition of a domain \( \Omega \) consisting of polygons in two dimensions or polyhedra in three dimensions satisfying a set of conditions specified in [11]. Denote by \( E_h \) the set of all edges or flat faces in \( \mathcal{T}_h \), and let \( E_h^0 = E_h \setminus \Gamma_+ \). Denote by \( h_T \) the diameter of \( T \) and mesh size \( h = \max_{T \in \mathcal{T}_h} h_T \) for \( \mathcal{T}_h \).

Denote by \( P_k(T) \) the set of polynomials on \( T \) with degree no more than \( k \). For a given integer \( k \geq 1 \), we define a finite element space as follows

\[
V_h = \{ v \in L^2(\Omega) : v|_T \in P_k(T), \forall T \in \mathcal{T}_h \}.
\]

Let \( e \in E_h^0 \setminus \Gamma_- \) shared by two elements \( T_1 \) and \( T_2 \), and \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) be unit outward normal vectors for \( T_1 \) and \( T_2 \) on \( e \), respectively. We define \( [\beta v \cdot \mathbf{n}] \) on \( e \) for \( v \in V_h \) as

\[
[\beta v \cdot \mathbf{n}] = \beta v|_{\partial T_1} \cdot \mathbf{n}_1 + \beta v|_{\partial T_2} \cdot \mathbf{n}_2.
\]

For \( e \in \partial T_1 \cap \Gamma_- \), let \( [\beta v \cdot \mathbf{n}] = \beta v|_{\partial T_1} \cdot \mathbf{n}_1 \).

We introduce a bilinear form \( a(\cdot, \cdot) \) as

\[
a(v, w) = \sum_{T \in \mathcal{T}_h} (\nabla \cdot (\beta v), \nabla \cdot (\beta w))_T + \sum_{e \in E_h^0} h_e^{-1} [\beta v \cdot \mathbf{n}] [\beta w \cdot \mathbf{n}] ds.
\]

Then the new finite element method is to find \( u_h \in V_h \) satisfying

\[
a(u_h, v) = (f, \nabla \cdot (\beta v)) + \sum_{e \in \Gamma_-} h_e^{-1} (\beta g \cdot \mathbf{n})(\beta v \cdot \mathbf{n}) ds, \quad \forall v \in V_h.
\]

We define a semi-norm \( \| \cdot \| \) as

\[
\| v \|^2 = a(v, v) = \sum_{T \in \mathcal{T}_h} \| \nabla \cdot (\beta v) \|^2_T + \sum_{e \in E_h^0} h_e^{-1} [\beta v \cdot \mathbf{n}]^2 ds.
\]
Lemma 2.1. The functional $|||\cdot|||$ defines a norm in $V_h$.

Proof. It suffices to check the positivity property for $|||\cdot|||$. Assume $v = 0$ for a given $v \in V_h$. Then we have

$$\sum_{T \in T_h} |||\nabla \cdot (\beta v)|||_T^2 + \sum_{e \in \mathcal{E}_h^0} h_e^{-1} |||[\beta v \cdot n]|||^2_e = 0,$$

which implies that $\nabla \cdot (\beta v) = 0$ on $T \in T_h$, $[\beta v \cdot n] = 0$ in $e \in \mathcal{E}_h^0 \setminus \Gamma_-$ and $\beta v \cdot n = 0$ on $e \in \Gamma_-$. Since $\beta \cdot n < 0$ on $\Gamma_-$, we have $v = 0$ on $\Gamma_-$, Thus $v$ is a solution of (1.1) with $f = g = 0$. The uniqueness of the solution of (1.1) implies $v = 0$. This completes the proof.

Lemma 2.2. The finite element scheme (2.1) has one and only one solution.

Proof. It suffices to prove the uniqueness. If $u_h^{(1)}$ and $u_h^{(2)}$ are two solutions of (2.1), then $e_h = u_h^{(1)} - u_h^{(2)}$ would satisfy the following equation

$$a(e_h, v) = 0, \quad \forall v \in V_h.$$

Letting $v = e_h$ in the above equation gives

$$|||e_h|||^2 = a(e_h, e_h) = 0,$$

which implies $e_h \equiv 0$. We have completed the proof of the lemma.

3. Error Estimate. In this section, we will estimate the difference between the true solution $u$ and its finite element approximation $u_h$ from (2.1).

For any function $\varphi \in H^1(T)$, the following trace inequality holds true (see [11] for details):

$$|||\varphi|||^2_e \leq C \left( \frac{1}{h_T} |||\varphi|||_T^2 + h_T |||\nabla \varphi|||_T^2 \right).$$

Lemma 3.1. The bilinear form $a(\cdot, \cdot)$ satisfies the following continuity property.

$$a(v, w) \leq |||v||| |||w|||.$$

Proof. It follows from the Cauchy-Schwarz inequality,

$$a(v, w) = \sum_{T \in T_h} (\nabla \cdot (\beta v), \nabla \cdot (\beta w))_T$$

$$+ \sum_{e \in \mathcal{E}_h^0} h_e^{-1} |||[\beta v \cdot n]|||^2_e |||[\beta w \cdot n]|||^2_e$$

$$\leq \left( \sum_{T \in T_h} |||\nabla \cdot (\beta v)|||_T^2 \right)^{1/2} \left( \sum_{T \in T_h} |||\nabla \cdot (\beta w)|||_T^2 \right)^{1/2}$$

$$+ \left( \sum_{e \in \mathcal{E}_h^0} h_e^{-1} |||[\beta v \cdot n]|||^2_e \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h^0} h_e^{-1} |||[\beta w \cdot n]|||^2_e \right)^{1/2}$$

$$\leq C |||v||| |||w|||.$$
which finishes the proof. □

Define

$$\|u\|_{k+1,T_h}^2 = \sum_{T \in T_h} \|u\|_{k+1,T}^2.$$

**Lemma 3.2.** Let $u$ be the solution of (1.1) and $Q_h u \in V_h$ be the $L^2$ projection of $u$ defined element by element. Then we have

$$\|u - Q_h u\| \leq C h^k \|u\|_{k+1,T_h}. \tag{3.3}$$

**Proof.** Using the definition of $Q_h$ and (3.1), we have

$$\|u - Q_h u\|^2 = \sum_{T \in T_h} \|\nabla \cdot (\beta(u - Q_h u))\|_T^2 + \sum_{e \in \Gamma_{-}} \int_{e} h_e^{-1} \|\beta(u - Q_h u) \cdot n\|_{e}^2$$

$$\leq C \sum_{T \in T_h} (\|\nabla (u - Q_h u)\|_T^2 + h^{-2} \|u - Q_h u\|_T^2)$$

$$\leq C h^{2k} \|u\|^2_{k+1,T_h}.$$

We have proved the lemma. □

**Theorem 3.3.** Let $u_h \in V_h$ be the finite element solution of the problem (1.1) arising from (2.1). Then there exists a constant $C$ such that

$$\|u - u_h\| \leq C h^k \|u\|_{k+1,T_h}. \tag{3.4}$$

**Proof.** Obviously, the true solution $u$ of (1.1) satisfies

$$a(u, v) = (f, \nabla \cdot (\beta v)) + \sum_{e \in \Gamma_{-}} \int_{e} h_e^{-1} (\beta g \cdot n)(\beta v \cdot n) ds, \quad \forall v \in V_h.$$

Subtracting (2.1) from the above equation implies

$$a(u - u_h, v) = 0, \quad \forall v \in V_h. \tag{3.5}$$

By adding and subtracting $Q_h u$ in (3.5), we have

$$a(Q_h u - u_h, v) = -a(u - Q_h u, v), \quad \forall v \in V_h.$$

Letting $v = Q_h u - u_h$ in the above equation and using (3.5), (3.2) and (3.3), we arrive

$$\|Q_h u - u_h\|^2 = a(Q_h u - u_h, Q_h u - u_h)$$

$$= \|a(u - Q_h u, Q_h u - u_h)\|$$

$$\leq \|u - Q_h u\| \|Q_h u - u_h\|$$

$$\leq C h^k \|u\|_{k+1,T_h} \|Q_h u - u_h\|,$$

which implies

$$\|Q_h u - u_h\| \leq C h^k \|u\|_{k+1,T_h}. \tag{3.6}$$

Using the triangle inequality, (3.3) and (3.6), we have

$$\|u - u_h\| \leq \|u - Q_h u\| + \|Q_h u - u_h\| \leq C h^k \|u\|_{k+1,T_h}.$$

The proof of the theorem is completed. □

4.1. Test 1. Let $\Omega = (0, 1) \times (0, 1)$. In this test, the constant convection vector is given by $\beta = (1, 1)$, and the exact solution is chosen as

$$u(x, y) = \sin(\pi x) \sin(\pi y).$$

In this problem, the inflow boundary is chosen as $\Gamma_- = \{x = 0, y \in (0, 1)\} \cup \{y = 0, x \in (0, 1)\}$. Homogeneous Dirichlet boundary condition and proper function $f$ are chosen such that $u$ is the exact solution to (1.1). We shall execute the algorithm (2.1) on the uniform triangular/rectangular/deformed rectangular meshes to validate our theoretical conclusions. Denote the mesh size by $h$. Figure 4.1 shows three different meshes used for numerical experiment. For the deformed rectangular mesh, Figure 4.1 (c) (denoted as mesh level 1) shows the initial mesh, and the next level of mesh is derived by connecting the middle point of each element with the middle point of corresponding edge.

![Fig. 4.1. (a) Uniform triangular mesh with size $h = 1/4$; (b) Uniform rectangular mesh with size $h = 1/4$; (c) Deformed rectangular mesh level 1.](image)

Table 4.1 Test 1: Error profile and convergence rate for $k = 1$ on uniform triangular mesh.

<table>
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<th>$| u - u_h |$</th>
<th>rate</th>
<th>$| u - u_h |$</th>
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<td>1.81</td>
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The error profiles and convergence rates are shown in Table 4.1-4.3. It shows that the error measured in $\| \cdot \|$-norm converges with first order. The error measured in $\| \cdot \|$-norm has the order $O(h^{1.8})$.

4.2. Test 2. In the second test, we shall use the same problem setting as that in Test 1. In this problem, the mesh is also chosen as uniform triangular mesh as shown in Figure 4.1 (a). We shall choose different convection vector as $\beta = (1, -1)$. In this problem, the inflow boundary $\Gamma_- = \{x = 0, y \in (0, 1)\} \cup \{y = 1, x \in (0, 1)\}$. In this case, the mesh is aligned with the convection vector.
Table 4.2
Test 1: Error profile and convergence rate for \( k = 1 \) on uniform rectangular mesh.

| \( h \)  | \( |||u - u_h||| \) rate | \( ||u - u_h||| \) rate |
|--------|--------------------------|--------------------------|
| 1/4    | 8.5739e-01               | 1.0630e-01               |
| 1/8    | 4.7326e-01, 0.86         | 5.4570e-02, 0.96         |
| 1/16   | 2.2541e-01, 1.07         | 2.1218e-02, 1.36         |
| 1/32   | 1.0298e-01, 1.13         | 6.8941e-03, 1.62         |
| 1/64   | 4.8150e-02, 1.10         | 2.0373e-03, 1.76         |
| 1/128  | 2.3217e-02, 1.05         | 5.7546e-04, 1.82         |

Table 4.3
Test 1: Error profile and convergence rate for \( k = 1 \) on deformed rectangular mesh.

| \( h \)     | \( |||u - u_h||| \) rate | \( ||u - u_h||| \) rate |
|------------|--------------------------|--------------------------|
| Mesh Level 1 | 8.6932e-01               | 1.0628e-01               |
| Mesh Level 2 | 4.6769e-01, 0.89         | 5.1066e-02, 1.06         |
| Mesh Level 3 | 2.2206e-01, 1.07         | 1.9042e-02, 1.42         |
| Mesh Level 4 | 1.0309e-01, 1.11         | 5.9677e-03, 1.67         |
| Mesh Level 5 | 4.9175e-02, 1.07         | 1.7070e-03, 1.81         |
| Mesh Level 6 | 2.4054e-02, 1.03         | 4.6799e-04, 1.87         |

Table 4.4 shows error profiles and convergence rate. It can be seen that the errors measured in \( |||\cdot||| \)-norm and \( ||\cdot|| |\)-norm converge with order \( O(h^2) \) and \( O(h^2) \) respectively.

Fig. 4.2. Numerical solutions for Test 3 in (a) and for Test 4 in (b).

In the following tests, we shall use uniform triangular mesh for testing.

4.3. Test 3. Let \( \Omega = (0, 1)^2 \) and the constant convection vector \( \beta = (\cos \theta, \sin \theta) = (b_1, b_2) \) with \( \theta = \frac{\pi}{6} \). The analytical solution is chosen as

\[
u(x, y) = \frac{1}{(y - \frac{b_2}{b_1} x - \frac{1}{2})^2 + \frac{1}{10}}
\]

It can be checked from (1.1) that \( f = 0 \) for the given convection vector \( \beta \) and exact solution \( u(x, y) \). It is known that \( u \in H^2(\Omega) \).
<table>
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<td>Test 2: Error profile and convergence rate for $k = 1$.</td>
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Error profiles and convergence rate are shown in Table 4.5. Similar results of $\|\cdot\|$-norm can be observed. The error of $\|\cdot\|$-norm has convergence rate $O(h^{1.3})$. The contour plotting is shown in Figure 4.3 (a), which demonstrates that the contour of the numerical solution is parallel to the convection vector.

![Fig. 4.3. Contours of numerical solutions for Test 3 in (a) and for Test 4 in (b). Arrows plot the direction of streamlines.](image)

### 4.4 Test 4.

Let $\Omega = (0, 1)^2$, $\beta = (\cos \theta, \sin \theta) = (b_1, b_2)$ with $\theta = \frac{\pi}{6}$, and $u$ is chosen as follows:

$$u(x, y) = \begin{cases} 
\frac{1}{(y - \frac{b_2}{b_1}x - \frac{1}{2})^2 + \frac{1}{100}}, & \text{if } y \geq \frac{b_2}{b_1}x \\
\frac{20}{7}, & \text{if } y < \frac{b_2}{b_1}x 
\end{cases}$$
The exact solution $u$ is $H^{1+\epsilon}(\Omega)$.  

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Although in this problem, the solution $u \notin H^2(\Omega)$ but only $H^{1+\epsilon}(\Omega)$, we cannot apply our theorem to predict the convergence results. Error profiles and convergence rate are shown in Table 4.6. The numerical solution of the scheme (2.1) can approximate the exact solution well. One can observe that the convergence rate of $\|\cdot\|$-norm and $\|\cdot\|$-norm are $O(h^{0.8})$ and $O(h^{1.3})$ respectively. The contour plotting is shown in Figure 4.3 (b), which demonstrates that the contour of the numerical solution is parallel to the convection vector.

4.5. **Test 5.** In this test, the constant convection vector is given as $\beta = [1, \tan(35^\circ)]$, the inflow boundary is $\Gamma_- = \{0\} \times \{(0, 1)\} \cup \{(0, 1)\} \times \{0\}$, and the boundary data is:

$$g(x, y) = \begin{cases} 
2 & \text{if } x = 0, \ y \in (0, 1), \\
1 & \text{if } y = 0, \ x \in (0, 1).
\end{cases}$$

The streamline for this test is $y = \tan(35^\circ)x$ and the exact solution is two constants with jump along streamline.

In this test, the uniform triangulation has been used in the numerical experiment. In this test, the triangulation is not aligned with the convection vector. The numerical solution for Test 5 is plotted in Figure 4.4 (Left). The plots show that our solutions are free of oscillations. Figure 4.5 numerical solutions with respect to different meshes. As one can observe that, the numerical simulation on finer mesh produces better approximation since more smearing is observed in the numerical solution on coarser mesh.

4.6. **Test 6.** In this test, the convection vector is chosen as $\beta = [-1, \tan(35^\circ)]$. In this case, the inflow boundary is $\Gamma_- = \{1\} \times \{(0, 1)\} \cup \{(0, 1)\} \times \{0\}$ and the inflow boundary condition is:

$$g(x, y) = \begin{cases} 
2 & \text{if } x = 1, \ y \in (0, 1), \\
1 & \text{if } y = 0, \ x \in (0, 1).
\end{cases}$$

The streamline for this test is $y = -\tan(35^\circ)x$ and the exact solution is two constants with jump along streamline.

In this test, the convection vector has different direction as that of Test 5. Also, the triangulation is not aligned with the convection vector. The numerical solution for Test 6 is plotted in Figure 4.4 (Right). Again, the plots show that our solutions are free of oscillations.
Fig. 4.4. Plots of numerical solutions, contours, and outflow profile at \( x = 1 \) and \( x = 0 \) for Test 5 (Left figures); Test 6 (Right figures).

4.7. Test 7. Here \( \beta = (-y, x) \), the inflow boundary is \( \Gamma_- = \{1\} \times \{(0,1)\} \cup \{(0,1)\} \times \{0\} \) and the inflow boundary condition is:

\[
g(x, y) = \begin{cases} 
-1 & \text{if } y = 0, \ 0 < x < 43/64, \\
1 & \text{if } y = 0, \ 43/64 \leq x < 1, \\
1 & \text{if } x = 1, \ y \in (0,1).
\end{cases}
\]

In this test, the convection vector is a function valued vector, which is a quarter of a circle centered at origin. The numerical solution for Test 7 is plotted in Figure 4.6 (Left). The contour plot shown in the figure shows a fairly good alignment of
convection vector. One can observe no oscillations through the horizontal profile plot although the smearing can be seen in the plot.

4.8. Test 8. Here $\beta = (y, 1 - x)$, the inflow boundary is $\Gamma_- = \{1\} \times \{(0, 1)\} \cup \{(0, 1)\} \times \{0\}$ and the inflow boundary condition is:

$$g(x, y) = \begin{cases} 
1 & \text{if } y = 0, \ 0 < x < 21/64, \\
-1 & \text{if } y = 0, \ 21/64 \leq x < 1, \\
1 & \text{if } x = 0, \ y \in (0, 1).
\end{cases}$$

The convection vector in this test is a quarter of circle centered at $(1, 0)$. The numerical solution for Test 8 is plotted in Figure 4.6 (Right), and again the plot is also free of oscillations. Similarly, one can observed that the contours of the numerical solution align with the convection vector $\beta = (y, -x)$.

4.9. Test 9. Here $\beta = (y, 0.5 - x)$, the inflow boundary is $\Gamma_- = \{1\} \times \{(0, 1)\} \cup \{(0, 1)\} \times \{0\}$ and the inflow boundary condition is:

$$g(x, y) = \begin{cases} 
0 & \text{if } x = 0, \ 0 \leq y \leq 1, \\
0 & \text{if } y = 1, \ 0.5 \leq x \leq 1, \\
0 & \text{if } y = 0, \ 0 \leq x \leq 0.17, \\
1 & \text{if } y = 0, \ 0.17 \leq x \leq 0.33, \\
0 & \text{if } y = 0, \ 0.33 \leq x < 0.5.
\end{cases}$$

The convection vector in this test is half of a circle centered at $(0.5, 0)$. The numerical solutions to Test 9 are plotted in Figure 4.7. Similar conclusion can be obtained.

REFERENCES

Fig. 4.6. Plots of numerical solutions, contours, and horizontal profile at $y = 0.5$ for Test 7 (Left figures); Test 8 (Right figures).


[10] H. Sterck, T. Manteuffel, S. McCormick and L. Olson, Least-squares finite element methods and
Fig. 4.7. Plots of numerical solutions (Left), contours (Right) for Test 9.
