A WEAK GALERKIN GENERALIZED MULTISCALE FINITE
ELEMENT METHOD
LIN MU*, JUNPING WANG†, AND XIU YE‡

Abstract. In this paper, we propose a general framework for weak Galerkin generalized multiscale (WG-GMS) finite element method for the elliptic problems with rapidly oscillating or high contrast coefficients. This general WG-GMS method features in high order accuracy on general meshes and can work with multiscale basis derived by different numerical schemes. A special case is studied under this WG-GMS framework in which the multiscale basis functions are obtained by solving local problem with the weak Galerkin finite element method. Convergence analysis and numerical experiments are obtained for the special case.

Key words. weak Galerkin, multiscale, finite element methods, elliptic problems with rapidly oscillating or high contrast coefficients, polyhedral meshes

AMS subject classifications. Primary: 65N15, 65N30; Secondary: 35J50

1. Introduction. Many problems arising from science and engineering have features at multiple scales such as physical processes in strongly heterogeneous media including deformation or diffusion in composite materials and flow in porous medium. For the problems with multiple scales and high contrast, the convergence of the standard finite element methods requires a mesh size \( h \) small enough to resolve the fine scale size. The cost of computations is often prohibitively expensive. Multiscale finite element methods are designed to solve multiscale problems efficiently that have the dimension of the coarse grid and resolve the fine scale features through construction of multiscale basis. The development of multiscale finite element methods is an active research field with significant study over the past decade. Multiscale techniques have been applied to different finite element methods for solving elliptic problems, for continuous finite element methods [3, 5, 6, 11, 16, 18, 19], for mixed finite element methods [1, 2, 7, 9], for discontinuous Galerkin methods [4].

The studies in [15, 16] show that the accuracy of multiscale finite element methods is sensitive to the boundary conditions imposed on computing basis functions. The imposed boundary conditions should have similar oscillatory behavior as the fine-scale solution. Taking this into consideration, generalized multiscale finite element methods construct additional multiscale basis functions by solving a carefully designed local spectral problem to enrich the initial multiscale space [8, 12, 13, 14, 17]. Specially, the generalized multiscale finite element method works well for more general problems such as materials with non-periodic properties, non-separable scales, and random coefficients.

Weak Galerkin method is a newly developed general finite element technique for solving partial differential equations (PDE). The WG methods, by design, make use

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*Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN, 37831 (mul1@ornl.gov).
†Division of Mathematical Sciences, National Science Foundation, Arlington, VA 22230 (jwang@nsf.gov). The research of Wang was supported by the NSF IR/D program, while working at National Science Foundation. However, any opinion, finding, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.
‡Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204 (xxye@ualr.edu). This research was supported in part by National Science Foundation Grant DMS-1115097.
of discontinuous piecewise polynomials and can be considered as a natural extension of the standard Galerkin finite element methods for functions with discontinuity. Therefore the WG method has the flexibility of using discontinuous elements and the simplicity in formulation of using continuous elements. In summary, the WG methods have the following features:

1. The WG method is highly flexible in which high order approximations and finite element partitions with arbitrary shape of polygons and polyhedrons can be used.
2. The WG method has simple formulation which implies easy implementation and easy error analysis. The WG formulation can be derived by replacing standard derivatives by weakly-defined derivatives in the corresponding variational forms of the PDEs, with an option of adding a parameter independent stabilizer.

The key components of the WG methods are weak derivatives and parameter free stabilizers which are introduced to enforce the connection of discontinuous approximations between elements. The definitions of weak derivatives in the WG methods can be very general which should depend on the natures of the approximation functions. The general definition of weak derivative makes the WG method widely applicable and easy to fit into the frameworks of many numerical techniques such as multiscale finite element methods and least-squares finite element methods.

The weak Galerkin method was first introduced in [34]. Since then the WG methods have been successfully applied for solving the second order elliptic problems in [32, 33, 21, 20], for the biharmonic equations in [22, 24, 26] and for the Brinkman equations [28]. The WG methods have also been used for the elliptic interface problems [23], for the Helmholtz equation [27, 25], for the Maxwell equations [30] and for the Stokes equations [35].

In this paper, we will develop a weak Galerkin generalized multiscale finite element framework for the elliptic problem with rapidly oscillating and high contrast coefficients. Weakly defined derivative makes the WG-GMS methods compatible with multiscale basis derived by different numerical schemes such as finite element methods, continuous or discontinuous; spectral methods or others. A special case under the WG-GMS framework is investigated. In this case study, the weak Galerkin finite element method is used to compute local snapshots. The numerical results demonstrate the accuracy of the WG-GMS finite element solutions after significant dimension reductions.

We consider the second order elliptic equation with rapidly oscillating or high contrast coefficients that seeks an unknown function \( u \) satisfying

\[
\begin{align*}
-\nabla \cdot (a \nabla u) &= f, & \text{in } \Omega, \\
u &= g, & \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a polytopal domain in \( \mathbb{R}^d \) (polygonal or polyhedral domain for \( d = 2, 3 \)), \( \nabla u \) denotes the gradient of the function \( u \), and \( a \) is a symmetric \( d \times d \) matrix-valued function in \( \Omega \). We shall assume that there exists a positive number \( \lambda > 0 \) such that

\[
\xi^t a \xi \geq \lambda \xi^t \xi, \quad \forall \xi \in \mathbb{R}^d.
\]

Here \( \xi \) is understood as a column vector and \( \xi^t \) is the transpose of \( \xi \).

2. Framework of the WG-GMS Finite Element Method. Let \( T_H \) be a partition of a domain \( \Omega \) consisting of polygons in two dimensions or polyhedra in
three dimensions satisfying a set of conditions specified in [33]. Denote by $\mathcal{E}_H$ the set of all edges or flat faces in $T_H$. Let $\mathcal{E}_H^0 = \mathcal{E}_H \setminus \partial \Omega$ and $\mathcal{E}_H^b = \mathcal{E}_H \setminus \mathcal{E}_H^0$.

Let $\Psi_i$ ($i = 1, \cdots, N$) be multiscale basis functions which are derived by solving local problems on each $K \in T_H$. Define a WG-GMS finite element space $V_H$ as

\[(2.1) \quad V_H = \text{span}\{\Psi_1, \cdots, \Psi_N\}.\]

Define a subspace of $V_H$,

\[(2.2) \quad V_H^0 = \{v : v \in V_H, v = 0 \text{ on } \partial \Omega\}.\]

**Definition 2.1.** For any $v \in V_H$ and $K \in T_H$, let $\nabla_w v$ denote a weak gradient of $v$ on $K$ which should possess the following properties:

- **P1:** $\nabla_w v$ is a good approximation of $\nabla v$ on $K$.
- **P2:** $\nabla_w v$ can be computed effectively.

**Definition 2.2.** For any $v, w \in V_H$, we define a stabilizer $s(v, w)$ that has the following properties:

- **P3:** $s(v, w)$ is symmetric, semi-positive definite and parameter independent.
- **P4:** $s(v, w) = 0$ if $v$ or $w$ is continuous.

Let $Q_b g$ be an approximation of $g$. Then we have the WG-GMS finite element method stated as follows.

**Algorithm 1.** (The WG-GMS Finite Element Method) A numerical approximation for (1.1) and (1.2) can be obtained by seeking $u_H \in V_H$ satisfying both $u_H = Q_b g$ on $\partial \Omega$ and the following equation:

\[(2.3) \quad (\nabla_w u_H, \nabla_w v) + s(u_H, v) = (f, v), \quad \forall \ v \in V_H^0.\]

**Remark 2.1.** The definitions of the weak gradient $\nabla_w$ and the stabilizer $s(\cdot, \cdot)$ in (2.3) are general that can be defined in many different ways. This makes the WG-GMS method (2.3) highly flexible.

The multiscale basis functions $\Psi_i$ in (2.1) can be found by solving the following local problem on each $K \in T_H$,

\[(2.4) \quad -\nabla \cdot (a \nabla \Psi_i) = 0, \quad \text{in } K,\]

\[(2.5) \quad \Psi_i = \Phi_i, \quad \text{on } \partial K,\]

where $\Phi_i$ is a prescribed function.

Different numerical schemes can be used to compute the local snapshots by solving the local problem (2.4)-(2.5). Definitions of weak gradient $\nabla_w$ and stabilizer $s(\cdot, \cdot)$ will depend on how the WG-GMS basis functions are computed.

**3. Case Study: Finding the Multiscale Basis by the WG Method.** In this case study, the WG-GMS finite element method (2.3) will be implemented through the following steps:

1. Use the WG method to compute local snapshots by solving the local problem (2.4)-(2.5).
2. Reduce the dimension of the snapshot space by solving local eigenvalue problems.
3. Use the reduced basis to form an offline space $V_H$.
4. Define weak gradient $\nabla_w$ and stabilizer $s(\cdot,\cdot)$ for the basis of the snapshot space.
5. Obtain the WG-GMS finite element solution by solving the system (2.3).
6. Analyze the convergence rate for the WG-GMS finite element solution.

The details of the implementation of the WG-GMS method will be discussed in the following subsections.

### 3.1. Construction of the Basis for the Snapshot Space.

In this section, we will use the weak Galerkin method to compute the local snapshots by solving the local problem (2.4)-(2.5).

First let $\mathcal{T}_h$ be a fine grid partition, a refinement of the coarse grid $\mathcal{T}_H$. Without loss of generality, we assume $d = 2$ for simplicity. Denote by $\mathcal{T}_h(K)$ the restriction of the fine mesh $\mathcal{T}_h$ on $K \in \mathcal{T}_H$. Obviously, we have $\mathcal{T}_h = \bigcup_{K \in \mathcal{T}_H} \mathcal{T}_h(K)$. For a given integer $k \geq 1$, let $V_h(K)$ be the weak Galerkin finite element space associated with $\mathcal{T}_h(K)$ defined as follows

$$
(3.1) \quad V_h(K) = \{ v = \{ v_0, v_b \} : v_0|_{T} \in P_k(T), \ v_b|_{e} \in P_k(e), \ e \in \partial T, \ T \in \mathcal{T}_h(K) \}
$$

and

$$
(3.2) \quad V^0_h(K) = \{ v : v \in V_h(K), \ v_b = 0 \text{ on } \partial K \}.
$$

For each $v \in V_h(K)$ and $T \in \mathcal{T}_h(K)$, we define a weak gradient of $v$ on $T$, denoted by $\nabla^T_w v$, as the unique polynomial $(\nabla^T_w v) \in [P_{k-1}(T)]^2$ satisfying the following equation,

$$
(3.3) \quad (\nabla^T_w v, q)_T = -(v_0, \nabla \cdot q)_T + (v_b, q \cdot n)_{\partial T}, \quad \forall q \in [P_{k-1}(T)]^2.
$$

After defining the weak gradient $\nabla^T_w$ in (3.3), we introduce two bilinear forms on $V_h(K)$ for $K \in \mathcal{T}_H$ as follows:

$$
(3.4) \quad s_K(v, w) = \rho \sum_{T \in \mathcal{T}_h(K)} h_T^{-1}(v_0 - v_b, w_0 - w_b)_{\partial T},
$$

$$
(3.5) \quad a_K(v, w) = \sum_{T \in \mathcal{T}_h(K)} (a\nabla^T_w v, \nabla^T_w w)_T + s_K(v, w),
$$

where $\rho$ can be any positive number. In practical computation, one might set $\rho = 1$.

Let $\mathcal{E}_h$ consists all the edges associated the fine grid $\mathcal{T}_h$. For each $e \in \mathcal{E}_h$, let $\Phi_{e,1}, \cdots, \Phi_{e,k+1}$ be the functions defined on $\mathcal{E}_h$ such that they are the $k + 1$ basis functions for $P_k(e)$ on $e$ and zero otherwise.

For each $e \in \mathcal{E}_h$, we can derive $k+1$ snapshots with $k$-th order accuracy by solving the following local problem.

**Algorithm 2.** Associate with each $e \in \mathcal{E}_h$, $k + 1$ weak Galerkin multiscale basis functions $\Psi_{e,j}$ can be calculated such that on each element $K \in \mathcal{T}_H$, $\Psi_{e,j} = \{ \Psi_0, \Psi_b \} \in V_h(K)$ satisfies both $\Psi_b = \Phi_{e,j}$ on $\partial K$ and the following equation:

$$
(3.6) \quad a_K(\Psi_{e,j}, v) = 0, \quad \forall v \in V^0_h(K).
$$
Then, we define snapshot space as
\begin{equation}
V^S_S = \text{span}\{\Psi_{e,i}, \ i = 1, \cdots, k+1, \ e \in \mathcal{E}_h\}.
\end{equation}

Let $E \in \mathcal{E}_H$ and $E = \cup e$ with $e \in \mathcal{E}_h$. Associated with each $e \subset E$, there are $k+1$ functions $\Psi_{e,i}, i = 1, \cdots, k+1$ based on (3.6). Define
\begin{equation}
V^H_H(E) = \text{span}\{\Psi_{e,1}, \cdots, \Psi_{e,k+1} \ e \subset E, \ e \in \mathcal{E}_h\}.
\end{equation}

\textbf{Remark 3.1.} If $E \in \mathcal{E}_H^0$, the nonzero support of any function in $V^H_H(E)$ consists two elements in $\mathcal{T}_H$ shared $E$ as shown in Figure 3.1 denoted by $\omega(E)$. It is also true that $\Psi$ is zero on $\partial \omega(E)$ for any $\Psi \in V^H_H(E)$ with $E \in \mathcal{E}_H^0$.

\subsection{Dimension Reduction.}
We will perform the basis reduction to obtain an offline space by following the framework of the generalized multiscale method in [17]. This can be achieved by determine the important modes among the multiscale basis $\Psi_{e,j}$ in the snapshot space $V^S_S$.

By using (3.7) and (3.8), it is easily to see that
\begin{equation}
V^S_S = \cup_{E \in \mathcal{E}_H} V^H_H(E) = \cup_{E \in \mathcal{E}_H^0} V^H_H(E) \oplus \cup_{E \in \mathcal{E}_H^b} V^H_H(E).
\end{equation}

Define
\begin{equation}
V^H_H = \cup_{E \in \mathcal{E}_H^0} V^H_H(E) \quad \text{and} \quad V^S_S = \cup_{E \in \mathcal{E}_H^b} V^H_H(E).
\end{equation}

In the following, we will reduce the number of the basis functions for $V^H_H(E_i)$ with $E_i \in \mathcal{E}_H^0$ to achieve the total basis reduction for the snapshot space $V^S_S$. This can be achieved by solving a local eigenvalue problem on each $\omega(E_i)$.

For convenience, we order the basis functions in $V^H_H(E_i)$ as
\begin{equation}
V^H_H(E_i) = \text{span}\{\Psi_{i,1}, \cdots, \Psi_{i,J_i}\}
\end{equation}
where $J_i = l_i \times (k+1)$ is the dimension of $V^H_H(E_i)$ with $l_i$ being the number of the fine grid edges $e$ in $E_i$.

Associated each $E_i \in \mathcal{E}_H^0$, we will solve an eigenvalue problem on $\omega(E_i)$: find $\lambda_i$ and $Z_i \in V^H_H(E_i)$ such that
\begin{equation}
\lambda_i a_i(Z_i, w) = s_i(Z_i, w), \ \forall w \in V^H_H(E_i),
\end{equation}
where
\begin{equation*}
a_i(v, w) = \sum_{K \in \omega(E_i)} a_K(v, w),
\end{equation*}
\begin{equation*}
s_i(v, w) = \sum_{K \in \omega(E_i)} s_K(v, w).
\end{equation*}
Let eigenvalue $\lambda^j_i$ be the solution of the equation (3.10) with the order
$$\lambda^1_i < \lambda^2_i < \cdots < \lambda^j_i,$$
and $\{Z^1_i, Z^2_i, \cdots, Z^j_i\}$ be the corresponding eigenvectors with $Z^j_i = (z^j_{i,m})^T_{m=1}.$
Define $\xi^i_j \in V_H(E_i)$ for $j = 1, \cdots, J_i,$
$$\xi^i_j = \sum_{m=1}^{J_i} z^j_{i,m} \Psi_{i,m}.$$ 
It is easy to see that
$$V_H(E_i) = \text{span}\{\xi^1_j, \xi^2_j, \cdots, \xi^j_J\}.$$ 
Now we define $V_H^i(E_i) \subset V_H(E_i)$ formed by the first $M_i$ basis functions of $V_H(E_i)$ in (3.11) with $M_i \leq J_i$, i.e.
$$V_H^i(E_i) = \text{span}\{\xi^1_j, \xi^2_j, \cdots, \xi^i_{M_i}\}, \quad \forall E_i \in \mathcal{E}_H^0.$$ 
Define $V_H^0$ as
$$V_H^0 = \bigcup_{E \in \mathcal{E}_H^0} V_H^i(E)$$
and $V_H$ as
$$V_H = V_H^0 \oplus V_H^b,$$
where $V_H^b$ is defined in (3.9).

### 3.3. Defining Weak Gradient $\nabla_w$ and Stabilizer $s(\cdot, \cdot)$

In order to implement the WG-GMS method (2.3), we need to define a weak gradient $\nabla_w$ and a stabilizer $s(\cdot, \cdot)$ for the functions in the snapshot space on a coarse grid. A good design of a weak gradient $\nabla_w$ should minimize the computational cost. In this case study, the multiscale basis functions $\Psi_{e,j}$ in (3.7) are the solution of the system (3.6). It implies that the multiscale basis functions $\Psi_{e,j}$ are piecewise polynomials on the fine mesh $\mathcal{T}_h$ and $\nabla^T_w \Psi_{e,j}$ has been calculated on each $T \in \mathcal{T}_h(K)$. Therefore we define a weak gradient $\nabla_w$ of a basis function $\Psi_{e,j} \in V_H^S$ on a coarse grid $K$ as follows:
$$\nabla_w \Psi_{e,j}|_T = \nabla^T_w \Psi_{e,j}, \quad \forall T \in \mathcal{T}_h(K).$$

Then patching them together, we derive $\nabla_w \Psi_{e,j}$ on a coarse grid $K$.

The definition of $\nabla_w$ in (3.12) on coarse grid $K$ is well defined based on the Definition 2.1. It is a good approximation of the standard gradient on $K$ and also the cheapest since it is already computed in (3.6).

Now we define a stabilizer $s(\cdot, \cdot)$ as
$$s(v, w) = \cup_{K \in \mathcal{T}_H} s_K(v, w), \quad \forall v, w \in V_H^S.$$ 
Obviously, $s(v, w)$ is symmetric, semi-positive definite and parameter independent.

There is no additional cost to compute $s(v, w)$ since $s_K(v, w)$ has been calculated in (3.6). Now we can define a bilinear form $a(v, w)$ for $v, w \in V_H^S$ as
$$a(v, w) = \cup_{K \in \mathcal{T}_H} a_K(v, w), \quad \forall v, w \in V_H^S.$$ 
Let $Q_h$ be the $L^2$ projection from $L^2(e)$ onto $P_h(e)$ for $e \in \mathcal{E}_h$. Then for this special case, the WG-GMS finite element method is seeking $u_H \in V_H$ satisfying both $u_H = Q_h g$ on $\partial \Omega$ and the following equation:
$$a(u_H, v) = (f, v), \quad \forall v \in V_H^0.$$
3.4. Convergence. In this section we will derive the convergence analysis for the WG-GMS finite element solution $u_H$ in (3.15).

First let $\Psi_H = \{\Psi_0, \Psi_b\} \in V^S_H$ be the WG multiscale solution in snapshot space satisfying the both $\Psi_b = Q_{bg}$ on $\partial \Omega$ and the following equation:

$$a(\Psi_H, v) = (f, v), \quad \forall \ v \in V^S_{H, 0}. \quad (3.16)$$

Let $V_h$ be the weak Galerkin finite element space associated with the fine grid $T_h$ defined as follows

$$V_h = \{v = \{v_0, v_b\} : v_0|_T \in P_k(T), \ v_b|_e \in P_k(e), \ e \in \partial T, \ T \in T_h\}$$

and

$$V^0_h = \{v \in V_h : \ v_b = 0 \text{ on } \partial \Omega\}.$$

For any $v \in V_h$, let

$$||v|| := \sqrt{a(v, v)}.$$

Let $u_h = \{u_0, u_b\}$ be the weak Galerkin finite element solution on fine grid $T_h$ such that $u_b = Q_{bg}$ and

$$a(u_h, v) = (f, v), \quad \forall \ v \in V^0_h, \quad (3.17)$$

where $a(\cdot, \cdot)$ is defined in (3.14).

**Theorem 3.2.** Let $\Psi_H$ and $u_h$ be the solutions of the WG multiscale method (3.16) in snapshot space and of the WG method (3.17) on fine grid respectively. Then we have for $k = 1$,

$$\|u_h - \Psi_H\| \leq CH\|f\|. \quad (3.18)$$

**Proof.** It is not hard to see that $u_h - \Psi_H = 0$ on $\partial K$ for $K \in T_H$. Thus restricted on any $K \in T_H$, $u_h - \Psi_H \in V^0_h(K)$. The equation (3.6) and the fact $u_h - \Psi_H \in V^0_h(K)$ imply

$$a(\Psi_H, u_h - \Psi_H) = 0, \quad (3.19)$$

and

$$\|u_h - \Psi_H\|_K \leq CH\|u_h - \Psi_H\|_K. \quad (3.20)$$

Using (3.17) and (3.19), we have

$$\|u_h - \Psi_H\|^2 = a(u_h - \Psi_H, u_h - \Psi_H) = (f, u_h - \Psi_H).$$

Combining the inequality (3.20) and equation above, we have

$$\|u_h - \Psi_H\| \leq CH\|f\|,$$

which proves the theorem. ✷
Theorem 3.3. Let $u_h$ and $u_H$ be the solutions of the WG method (3.17) on the fine grid and of the WG-GMS method (3.15) respectively. The we have for $k = 1$

\[(3.21) \quad \|u_h - u_H\| \leq C(H\|f\| + \Lambda^{-1/2}\|\Psi_H\|),\]

where $\Lambda = \min_{i=1}^{M} \lambda^i_{M_i}$ with $M$ the number of $E \in \mathcal{E}_H^0$ and $\Psi_H$ is the solution of the equation (3.16).

Proof. Recall that $u_H$ is the solution of the WG-GMS method and $\Psi_H$ is the solution of (3.16) in the snapshot space. Since $V_H = V_H^0 \oplus V_H^b$ and $V_H^S = V_{H,0}^S \oplus V_H^b$, we can write $u_H = U_0 + U_g$ and $\Psi_H = \Psi_0 + \Psi_g$ with $\Psi_0 \in V_{H,0}^S$ and $U_0 \in V_H^0$ and $U_g, \Psi_g \in V_H^b$. Since $U_g$ and $\Psi_g$ are completely determined by the boundary condition, we have $U_g = \Psi_g$.

It follows from (3.11) that

\[\Psi_0 = \sum_{i=1}^{M} \sum_{j=1}^{J_i} c_{i,j} \xi_j.\]

Define $\Psi_0^r$ as,

\[(3.22) \quad \Psi_0^r = \sum_{i=1}^{M} \sum_{j=1}^{M_i} c_{i,j} \xi_j,\]

and $\Psi_H^r$ as

\[(3.23) \quad \Psi_H^r = \Psi_0^r + \Psi_g.\]

Since $V_H^0 \subset V_h^0$, we have

\[a(u_h, v) = (f, v), \quad \forall v \in V_H^0.\]

The difference of the above equation and (3.15) imply

\[(3.24) \quad a(u_h - u_H, v) = 0, \quad \forall v \in V_H^0.\]

It follows from (3.24) and $U_0, \Psi_0^r \in V_H^0$ that

\[
\begin{align*}
\|u_h - u_H\|^2 & = a(u_h - u_H, u_h - u_H) \\
& = a(u_h - u_H, u_h - (U_0 + U_g)) \\
& = a(u_h - u_H, u_h - (\Psi_0^r + \Psi_g)) \\
& = a(u_h - u_H, u_h - \Psi_H^r) \\
& = a(u_h - u_H, u_h - \Psi_H) + a(u_h - u_H, \Psi_H - \Psi_H^r) \\
& \leq C(\|u_h - \Psi_H\| + \|\Psi_H - \Psi_H^r\|) \|u_h - u_H\|.
\end{align*}
\]
It follows from (3.10) that
\[
\|\Psi_H - \Psi_H^r\|^2 = a(\Psi_H - \Psi_H^r, \Psi_H - \Psi_H^r)
\leq C \sum_{i=1}^{M} \sum_{K \in \omega(E_i)} a_i(\Psi_H - \Psi_H^r, \Psi_H - \Psi_H^r)
\leq C \sum_{i=1}^{M} \sum_{K \in \omega(E_i)} \lambda_i^{-1} s_i(\Psi_H - \Psi_H^r, \Psi_H - \Psi_H^r)
\leq C \sum_{i=1}^{M} \sum_{K \in \omega(E_i)} (\lambda_{M_i}^i)^{-1} a_i(\Psi_H - \Psi_H^r, \Psi_H - \Psi_H^r)
\leq C A^{-1} \sum_{i=1}^{M} \sum_{K \in \omega(E_i)} a_i(\Psi_H, \Psi_H)
\leq C A^{-1} \|\Psi_H\|^2,
\]
which implies
\[
(3.26) \quad \|\Psi_H - \Psi_H^r\| \leq C A^{-1/2} \|\Psi_H\|.
\]
Combining (3.26) and (3.18) with (3.25), we have
\[
\|u_h - u_H\| \leq C (H \|f\| + A^{-1/2} \|\Psi_H\|)
\]
which proves the theorem. □

Similarly, we can prove the following corollary.

**Corollary 3.4.** Let \(\Psi_H\) and \(u_H\) be the solutions of (3.16) in snapshot space and of the WG-GMS method (3.15) respectively. We have for \(k \geq 1\)
\[
(3.27) \quad \|\Psi_H - u_H\| \leq C A^{-1/2} \|\Psi_H\|,
\]
where \(A = \min_{i=1}^{M} \lambda_{M_i}^i\).

**4. Numerical Results.** In this section, we shall present several numerical examples based on numerical schemes (3.16) and (3.15) for validating the theoretical conclusions. The finite element space on the fine mesh of weak Galerkin approximation is chosen as follows:
\[
W_h = \{v = \{v_0, v_b\} : v_0|_T \in P_k(T), \ v_b|_e \in P_k(e), \ e \in \partial T, T \in T_h\}.
\]
4.1. Weak Galerkin Multiscale Approach. In this section, we will report the numerical performance of numerical scheme (3.16). The experiments are performed for linear element \( k = 1 \) and quadratic element \( k = 2 \), respectively. Two types of grids have been utilized in weak Galerkin simulation: the triangular grid and the rectangular grid.

Let \( \mathcal{T}_H \) and \( \mathcal{T}_h \) be the coarse partition and the fine partition of the domain \( \Omega \), which are plotted in Fig 4.1. First, the coarse rectangular mesh is constructed by dividing \( x \) and \( y \) direction into \( n_1 \) segments. Then, the corresponding fine mesh is generated on each coarse element by dividing \( x \) and \( y \) direction into \( n_2 \) segments. The triangular mesh is obtained by dividing the rectangular mesh by the negative slope. The size of coarse mesh \( \mathcal{T}_H \) is denoted by \( H = 1/n_1 \) and the size of fine mesh \( \mathcal{T}_h \) is denoted by \( h = 1/(n_1n_2) \). The snapshot space \( V_H^S \) is constructed by Algorithm 2, and then the weak Galerkin multiscale finite element solution is derived by numerical scheme (3.16).

The following two norms will be used in the tests:

Discrete \( H_1 \)-norm: \( \|v\| = \left( \sum_{T \in \mathcal{T}_h} (\|\nabla w v\|_T^2 + h^{-1} \|v_0 - v_b\|_{\partial T}^2) \right)^{1/2} \),

\( L^2 \)-norm: \( \|v\| = \left( \sum_{T \in \mathcal{T}_h} \|v_0\|^2 \right)^{1/2} \).

As the numerical analysis in Theorem 3.2, we expect a convergence rate of our weak Galerkin multiscale approximation as \( O(H^k) \) and \( O(H^{k+1}) \) for \( \|\cdot\| - \text{norm} \) and \( \|\cdot\| - \text{norm} \), respectively.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>( |u_h - \Psi_H| ) order</th>
<th>( |u_h - \Psi_H| ) order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear weak Galerkin element</td>
<td>Level 1</td>
<td>7.2215e-2</td>
</tr>
<tr>
<td></td>
<td>Level 2</td>
<td>2.0804e-2</td>
</tr>
<tr>
<td></td>
<td>Level 3</td>
<td>5.9682e-3</td>
</tr>
<tr>
<td></td>
<td>Level 4</td>
<td>1.6698e-3</td>
</tr>
<tr>
<td></td>
<td>Level 5</td>
<td>4.5799e-4</td>
</tr>
<tr>
<td></td>
<td>Level 6</td>
<td>1.2382e-4</td>
</tr>
<tr>
<td>Quadratic weak Galerkin element</td>
<td>Level 1</td>
<td>1.2218e-2</td>
</tr>
<tr>
<td></td>
<td>Level 2</td>
<td>2.1856e-3</td>
</tr>
<tr>
<td></td>
<td>Level 3</td>
<td>3.8978e-4</td>
</tr>
<tr>
<td></td>
<td>Level 4</td>
<td>6.9196e-5</td>
</tr>
<tr>
<td></td>
<td>Level 5</td>
<td>1.2256e-5</td>
</tr>
<tr>
<td></td>
<td>Level 6</td>
<td>2.1667e-6</td>
</tr>
</tbody>
</table>

4.1.1. Example 1. In the first example, let \( \Omega = (0,1)^2 \) and the exact solution is given as follows:

\[
(4.1) \quad u = \cos(\pi x) \cos(\pi y).
\]

\( f \) is chosen to satisfy the equation (1.1)-(1.2) with \( a = I \).
Table 4.2

Example 1. Convergence rate for rectangular mesh: Linear and quadratic weak Galerkin elements.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>$|u_h - \Psi_H|$</th>
<th>order</th>
<th>$|u_h - \Psi_H|$</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1</td>
<td>5.7355e-3</td>
<td></td>
<td>2.8439e-2</td>
<td></td>
</tr>
<tr>
<td>Level 2</td>
<td>1.3703e-3</td>
<td>2.0654</td>
<td>7.1055e-3</td>
<td>2.0009</td>
</tr>
<tr>
<td>Level 3</td>
<td>3.3820e-4</td>
<td>2.0185</td>
<td>1.7764e-3</td>
<td>2.0000</td>
</tr>
<tr>
<td>Level 4</td>
<td>8.4367e-5</td>
<td>2.0031</td>
<td>4.4140e-4</td>
<td>2.0000</td>
</tr>
<tr>
<td>Level 5</td>
<td>2.1086e-5</td>
<td>2.0004</td>
<td>1.1103e-4</td>
<td>1.9999</td>
</tr>
<tr>
<td>Level 6</td>
<td>5.2711e-6</td>
<td>2.0001</td>
<td>2.7757e-5</td>
<td>2.0000</td>
</tr>
</tbody>
</table>

We solve this problem by numerical scheme (3.16) with linear weak Galerkin element $k = 1$ and quadratic weak Galerkin element $k = 2$ for a sequence of meshes for testing errors and the convergence rates in both $\|\cdot\|$-norm and $\|\|\cdot\||$-norm. The initial mesh is chosen as $H = 1/2$ and $h = 1/8$, and then the next level of mesh is derived by refining the previous level of mesh. The numerical results, with respect to different mesh size for $k = 1$ and $k = 2$, are show in Table 4.1-4.2. On both rectangular mesh and triangular mesh, the superconvergence in $\|\|\cdot\||$-norm is observed. The convergence rate of $\|\cdot\|$-norm is shown as order $k+1$, which agrees with the theoretical conclusion.

Table 4.3

Example 2. Convergence rate: Linear weak Galerkin element on triangular mesh.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>$|u_h - \Psi_H|$</th>
<th>order</th>
<th>$|u_h - \Psi_H|$</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1</td>
<td>1.8862e-1</td>
<td></td>
<td>9.2008e-2</td>
<td></td>
</tr>
<tr>
<td>Level 2</td>
<td>6.6008e-2</td>
<td>1.5148</td>
<td>2.3531e-3</td>
<td>1.9672</td>
</tr>
<tr>
<td>Level 3</td>
<td>2.0422e-2</td>
<td>1.6925</td>
<td>5.9225e-4</td>
<td>1.9903</td>
</tr>
<tr>
<td>Level 4</td>
<td>5.8613e-3</td>
<td>1.8008</td>
<td>1.4835e-4</td>
<td>1.9972</td>
</tr>
<tr>
<td>Level 5</td>
<td>1.6275e-3</td>
<td>1.8486</td>
<td>3.7108e-5</td>
<td>1.9992</td>
</tr>
<tr>
<td>Level 6</td>
<td>4.4349e-4</td>
<td>1.8757</td>
<td>9.2786e-6</td>
<td>1.9998</td>
</tr>
</tbody>
</table>

Table 4.4

Example 2. Convergence rate: Linear weak Galerkin element on rectangular mesh.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>$|u_h - \Psi_H|$</th>
<th>order</th>
<th>$|u_h - \Psi_H|$</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1</td>
<td>4.7287e-1</td>
<td></td>
<td>7.7720e-2</td>
<td></td>
</tr>
<tr>
<td>Level 2</td>
<td>2.3862e-1</td>
<td>9.8673e-1</td>
<td>1.9813e-2</td>
<td>1.9718</td>
</tr>
<tr>
<td>Level 3</td>
<td>9.8767e-2</td>
<td>1.2726</td>
<td>4.9742e-3</td>
<td>1.9939</td>
</tr>
<tr>
<td>Level 4</td>
<td>3.3260e-2</td>
<td>1.5702</td>
<td>1.2445e-3</td>
<td>1.9989</td>
</tr>
<tr>
<td>Level 5</td>
<td>9.8222e-3</td>
<td>1.7597</td>
<td>3.1117e-4</td>
<td>1.9998</td>
</tr>
<tr>
<td>Level 6</td>
<td>2.7544e-3</td>
<td>1.8343</td>
<td>7.7796e-5</td>
<td>1.9999</td>
</tr>
</tbody>
</table>
4.1.2. Example 2. Let $\Omega = (0,1)^2$, the known analytic solution is given by

\[ u = x^3 y^4 + x^2 + \sin(xy) \cos(xy) \]  

and full tensor coefficient $a$ is shown as

\[ a = \begin{pmatrix} (x + 1)^2 + y^2 & \sin(xy) \\ \sin(xy) & (x + 1)^2 \end{pmatrix}. \]  

In this test, we conduct the linear weak Galerkin approximation for the elliptic equation with coefficient as full tensor. The calculation is performed on both triangular and rectangular grids.

Error profiles and convergence rates for this problem are given in Table 4.3-4.4. We observe that the convergence rates are as good as predicted by the theory. It is obtained optimal rate of convergence for $\| \cdot \|_{-}$-norm, which is shown as order $O(H^2)$. Again superconvergence for $\| \cdot \|_{-}$-norm is detected in the experiment.

<table>
<thead>
<tr>
<th>$1/h$</th>
<th>$| u_h - u |$</th>
<th>$| u_h - u |$</th>
<th>$| u_h - u |$</th>
<th>$| u_h - u |$</th>
<th>DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.5016e-1</td>
<td>4.8059e-1</td>
<td>2.6660e-1</td>
<td>4.9000e+1</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>2.8332e-2</td>
<td>2.4</td>
<td>1.4533e-1</td>
<td>1.7</td>
<td>2.2500e+2</td>
</tr>
<tr>
<td>32</td>
<td>6.8847e-4</td>
<td>5.4</td>
<td>1.0127e-2</td>
<td>3.8</td>
<td>7.0790e-3</td>
</tr>
<tr>
<td>64</td>
<td>1.5001e-4</td>
<td>2.2</td>
<td>4.1975e-4</td>
<td>4.6</td>
<td>7.0900e-3</td>
</tr>
<tr>
<td>128</td>
<td>3.7980e-5</td>
<td>2.0</td>
<td>1.0938e-4</td>
<td>1.9</td>
<td>4.6749e-4</td>
</tr>
<tr>
<td>256</td>
<td>9.4466e-6</td>
<td>2.0</td>
<td>2.7034e-5</td>
<td>2.0</td>
<td>9.1528e-5</td>
</tr>
<tr>
<td>512</td>
<td>2.3587e-6</td>
<td>2.0</td>
<td>6.7389e-6</td>
<td>2.0</td>
<td>2.2777e-5</td>
</tr>
</tbody>
</table>

4.1.3. Example 3. Let $\Omega = (0,1)^2$, the coefficient $a$ is set as follows

\[ a(x/\epsilon) = \frac{1}{4 + P(\sin(2\pi x/\epsilon) + \sin(2\pi y/\epsilon))}, \]  

where $P$ is a controlling parameter of the magnitude for the oscillation. We choose $P = 1.8$ for testing. The right hand side function $f(x,y)$ is chosen such that exact solution is given as

\[ u = \frac{\sqrt{4 - P^2}}{2} (x^2 + y^2). \]
Example 3. Convergence rate: Linear weak Galerkin multiscale approximation (3.16) on triangular mesh with fixed $H = 1/8$.

<table>
<thead>
<tr>
<th>$H/h$</th>
<th>$\epsilon = 0.4$</th>
<th>$\epsilon = 0.2$</th>
<th>$\epsilon = 0.2$</th>
<th>DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$|u - \Psi_H|$</td>
<td>order</td>
<td>$|u - \Psi_H|$</td>
<td>order</td>
</tr>
<tr>
<td>1</td>
<td>7.9596e-2</td>
<td>2.9976e-1</td>
<td>2.1338e-1</td>
<td>1.7600e+2</td>
</tr>
<tr>
<td>2</td>
<td>1.1960e-2</td>
<td>2.7889e-2</td>
<td>1.9594e-1</td>
<td>1.2e-1</td>
</tr>
<tr>
<td>4</td>
<td>1.0121e-3</td>
<td>3.6280e-3</td>
<td>5.5179e-2</td>
<td>1.8</td>
</tr>
<tr>
<td>8</td>
<td>2.5961e-4</td>
<td>2.0594e-4</td>
<td>2.8391e-3</td>
<td>4.3</td>
</tr>
<tr>
<td>16</td>
<td>6.4942e-5</td>
<td>2.0161e-4</td>
<td>3.8899e-4</td>
<td>3.1</td>
</tr>
<tr>
<td>32</td>
<td>1.6239e-5</td>
<td>2.040275e-5</td>
<td>9.3299e-5</td>
<td>1.9</td>
</tr>
<tr>
<td>64</td>
<td>4.0599e-6</td>
<td>2.010069e-5</td>
<td>2.3325e-5</td>
<td>2.0</td>
</tr>
</tbody>
</table>

The permeability $a$ is plotted in Figure 4.2 for $\epsilon = 0.4$, $\epsilon = 0.2$ and $\epsilon = 0.1$. We require that the size of mesh is fine enough in order to achieve the optimal rate of convergence and the desired accuracy. However, the finer mesh will increase the degree of freedom. In this example, we will compare the numerical results of fine mesh solution (3.17) and weak Galerkin multiscale finite element solution (3.16).

In Table 4.5, the weak Galerkin numerical results for $\epsilon = 0.4$, $\epsilon = 0.2$ and $\epsilon = 0.1$ is given, and the total degree of freedom (DOF) is shown in the last column of Table 4.5. It is noted that for smaller value of $\epsilon$, in order to achieve desired accuracy, the computing grid should be fine enough to resolve the information of coefficient $a$. For smaller value of $\epsilon$, the DOF needed for accurate simulation is increasing fast. In contrast, the same problem is tested by multiscale finite element methods. First, the snapshot space $V_{S}^H$ is constructed locally as Algorithm 2. Based on the snapshot space, the global approximation follows the numerical scheme (3.16). Here we choose coarse mesh with size $H = 1/8$ for testing. The numerical errors, convergence rates, and DOFs are shown in Table 4.6. By comparison of the DOF and errors, it is shown that the weak Galerkin multiscale methods can show the comparable accuracy with smaller size of DOF.

4.2. Weak Galerkin Generalized Multiscale Approach. In this section, we will present three numerical examples by using the weak Galerkin generalized multiscale approach (3.15). Let $\Omega = (0, 1) \times (0, 1)$, in the following three numerical examples we will solve the following second order elliptic equation:

$$ -\nabla \cdot (a \nabla) u = 1 \quad \text{in } \Omega $$

$$ u = 0 \quad \text{on } \partial \Omega. $$

(4.5)

4.2.1. Example 4. In this numerical example, the permeability field $a$ is depicted in Fig.4.3 (Left). In the simulation, let $n = 100$, we will use three different size of coarse mesh, which are $N = 5$, $N = 10$, and $N = 20$. Denote the fine grid solution as $u_h$ and the generalized multiscale weak Galerkin solution as $u_H$. The numerical error measured in relative $L^2$-norm are shown in Table 4.7. In these table, the term "dof per E" means the number of eigen-functions used for the generalized multiscale approximation on each of interior edge $E$. Notice that, in the case $n = 100$, $N = 5$, there are 40 basis functions of snapshot solution on each edge $E$. As we reducing the dimension of snapshot space, the generalized multiscale solution can still provide acceptable numerical solution. From Table 4.7, we observed clearly convergence of the method.
Example 4: Left: The permeability field $a$; Right: The weak Galerkin solution $u_h$ on the fine grid $100 \times 100$.

Table 4.7

<table>
<thead>
<tr>
<th>dof per $E$</th>
<th>$n = 100, N = 5$</th>
<th>$n = 100, N = 10$</th>
<th>$n = 100, N = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.7659e-01$</td>
<td>$4.9534e-02$</td>
<td>$1.3764e-02$</td>
</tr>
<tr>
<td>3</td>
<td>$1.7968e-01$</td>
<td>$4.2572e-02$</td>
<td>$4.3594e-03$</td>
</tr>
<tr>
<td>5</td>
<td>$1.7877e-01$</td>
<td>$5.3329e-03$</td>
<td>$1.8973e-03$</td>
</tr>
<tr>
<td>7</td>
<td>$1.7713e-01$</td>
<td>$4.6593e-03$</td>
<td>$1.7007e-03$</td>
</tr>
<tr>
<td>9</td>
<td>$1.7395e-01$</td>
<td>$4.3944e-03$</td>
<td>$1.4306e-03$</td>
</tr>
<tr>
<td>10</td>
<td>$1.6692e-01$</td>
<td>$4.3810e-03$</td>
<td>$9.8304e-04$</td>
</tr>
<tr>
<td>20</td>
<td>$1.6009e-02$</td>
<td>$4.1814e-03$</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>$1.5878e-02$</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

when adding more eigenfunctions to the offline space. However, the error will converge (column of Table 4.7) to a fixed error which is associate with the coarse-grid error. Furthermore, the coarse-grid error can be reduced by using smaller size of coarse-grid. In particular, when $N = 5$, the coarse-grid error is about $1.6e-02$; when $N = 10$, the coarse-grid error is reduced to about $4.2e-03$; and when $N = 30$, the coarse-grid error is reduced to about $1.0e-03$. Also, this confirms that the convergence rate of $L^2$-norm is $O(H^2)$.

In fact, the first 10 eigenfunctions for $N = 5$, the first 5 eigenfunctions for $N = 10$ are enough to achieve good performance in numerical simulation. Fig.4.3 (right) and Fig. 4.4 plot the fine solution $u_h$, the snapshot solution $\Psi_H$, and generalized multiscale solution $u_H$.

4.2.2 Example 5. In this experiment, the permeability field $a$ is depicted in Fig.4.5 (Left).

In the simulation, we still choose three different size of coarse meshes, which are $N = 5$, $N = 10$, $N = 20$. Table 4.8 gives the error profiles, from which we observe the similar performance as Example 4. As adding more eigenfunctions into the offline space, the error measured in $L^2$-norm converges to the fixed error. Moreover, the first
Example 4: The snapshot solution $\Psi_H$ (Left) and generalized multiscale solution $u_H$ (Right) on the coarse mesh $10 \times 10$.

Example 5: Left: The permeability field $a$; Right: The weak Galerkin solution $u_h$ on the fine grid $100 \times 100$.

7 eigenfunctions can provide acceptable numerical approximation. Fig.4.5 (right) and Fig. 4.6 plot the solution on the fine grid $u_h$, the solution in the snapshot space $\Psi_H$, and the generalized multiscale weak Galerkin solution $u_H$.

4.2.3. Example 6. In the last experiment, the permeability field $a$ is depicted in Fig.4.7 (Left).

In the simulation, we still choose three different size of coarse meshes, which are $N = 5$, $N = 10$, $N = 20$. Table 4.9 gives the error profiles, from which we observe the similar performance. As adding more eigenfunctions into the offline space, the error measured in $L^2$-norm converges to the fixed error. Moreover, the first 5 eigenfunctions can provide acceptable numerical approximation. Fig.4.7 (right) and Fig. 4.8 plot the solution on the fine grid $u_h$, the solution in the snapshot space $\Psi_H$, and the generalized multiscale weak Galerkin solution $u_H$. 
Table 4.8
Example 5: Error profiles for Fig.4.5 (Left).

<table>
<thead>
<tr>
<th>dof per $E$</th>
<th>$n = 100, N = 5$</th>
<th>$n = 100, N = 10$</th>
<th>$n = 100, N = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$|u_H - u_h|/|u_h|$</td>
<td>$|u_H - u_h|/|u_h|$</td>
<td>$|u_H - u_h|/|u_h|$</td>
</tr>
<tr>
<td>1</td>
<td>5.0244e-01</td>
<td>9.1201e-02</td>
<td>3.0496e-02</td>
</tr>
<tr>
<td>3</td>
<td>3.4537e-01</td>
<td>5.4538e-02</td>
<td>3.6105e-03</td>
</tr>
<tr>
<td>5</td>
<td>5.4634e-02</td>
<td>8.5770e-03</td>
<td>1.7102e-03</td>
</tr>
<tr>
<td>7</td>
<td>1.7630e-02</td>
<td>5.1387e-03</td>
<td>1.5940e-03</td>
</tr>
<tr>
<td>9</td>
<td>1.6787e-02</td>
<td>4.7343e-03</td>
<td>1.5506e-03</td>
</tr>
<tr>
<td>10</td>
<td>1.6618e-02</td>
<td>4.7118e-03</td>
<td>1.5452e-03</td>
</tr>
<tr>
<td>20</td>
<td>1.5928e-02</td>
<td>4.6273e-03</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>1.5853e-02</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Fig. 4.6. Example 5: The snapshot solution $\Psi_H$ (Left) and generalized multiscale solution $u_H$ (Right) on the coarse mesh $10 \times 10$.

REFERENCES


Example 6: Left: The permeability field $a$; Right: The weak Galerkin solution $u_h$ on the fine grid $100 \times 100$.

Table 4.9
Example 6: Error profiles for Fig.4.7 (Left).

<table>
<thead>
<tr>
<th>dof per</th>
<th>$n = 100, N = 5$</th>
<th>$n = 100, N = 10$</th>
<th>$n = 100, N = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$|u_H - u_h|/|u_h|$</td>
<td>$|u_H - u_h|/|u_h|$</td>
<td>$|u_H - u_h|/|u_h|$</td>
</tr>
<tr>
<td>1</td>
<td>8.0463e-01</td>
<td>4.4399e-01</td>
<td>4.7785e-02</td>
</tr>
<tr>
<td>3</td>
<td>5.2420e-01</td>
<td>1.6139e-01</td>
<td>3.3018e-02</td>
</tr>
<tr>
<td>5</td>
<td>2.9955e-01</td>
<td>7.6311e-02</td>
<td>2.9196e-02</td>
</tr>
<tr>
<td>7</td>
<td>2.3309e-01</td>
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Fig. 4.8. Example 6: The snapshot solution $u_h$ (Left) and generalized multiscale solution $u_{H}$ (Right) on the coarse mesh $10 \times 10$.