A FINITE VOLUME METHOD FOR THE NAVIER-STOKES PROBLEMS
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Abstract. We develop a finite volume method for solving the Navier-Stokes equations on triangular mesh. We prove that the unique solution of the finite volume method converges to the true solution with optimal order for velocity and for pressure in discrete $H^1$ norm and $L^2$ norm respectively.

Key words. finite volume methods, Navier-Stokes problems

AMS subject classifications. Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50

1. Introduction. Mass conservation is an important criteria in development of a numerical scheme for computational fluid dynamic. Finite volume methods are well known for their local conservativeness of the numerical fluxes. This is because the weak formulation of finite volume schemes for a partial differential equation is obtained by integrating the equation over a control volume. Due to its local conservativeness and simplicity, finite volume method is widely used in computational fluid mechanics and other applications. Finite volume methods have been investigated for convection-diffusion problems by many authors [1, 4, 7, 9, 12, 15, 18] and for the Stokes equations [2, 5, 6, 8, 11, 19]. In [13, 16], finite volume methods were used for the two-phase flow problems. A cell centered finite volume method is investigated in [10] for the Navier-Stokes equations.

The goal of this paper is to develop a vertex centered finite volume method for the Navier-Stokes equations on triangular mesh. We will prove existence and uniqueness of the method in Section 3. Optimal order error analysis will be derived in Section 4 and numerical experiments will be conducted in Section 5.

We consider the Navier-Stokes equations

\begin{align}
-\nu \Delta u + u \cdot \nabla u + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
\nu \varepsilon = 0 & \quad \text{on } \partial \Omega,
\end{align}

where the symbols $\Delta$, $\nabla$, and $\nabla \cdot$ denote the Laplacian, gradient, and divergence operators, respectively, and $f(x)$ is the external volumetric force acting on the fluid at $x \in \Omega \subset \mathbb{R}^2$.

We use standard definitions for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot, \cdot)_{s,D}$, norms $\| \cdot \|_{s,D}$, and seminorms $|\cdot|_{s,D}$ for $s \geq 0$. For example, for any integer $s \geq 0$, the seminorm $|\cdot|_{s,D}$ is given by

$$
|v|_{s,D} = \left( \sum_{|\alpha|=s} \int_D |\partial^\alpha v|^2 dD \right)^{\frac{1}{2}},
$$

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Fig. 2.1. Primal partition and control volume

with the usual notation
\[ \alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2, \quad \partial^\alpha = \partial^\alpha_1 \partial^\alpha_2. \]

The Sobolev norm \( \| \cdot \|_{m,D} \) is given by
\[
\|v\|_{m,D} = \left( \sum_{j=0}^{m} |v|_{j,D}^2 \right)^{\frac{1}{2}}.
\]

The space \( H^0(D) \) coincides with \( L^2(D) \), for which the norm and the inner product are denoted by \( \| \cdot \|_D \) and \( (\cdot, \cdot)_D \), respectively. When \( D = \Omega \), we shall drop the subscript \( D \) in the norm and inner product notation. We also use \( L^2_0(\Omega) \) to denote the subspace of \( L^2(\Omega) \) consisting of functions with mean value zero.

2. Finite volume method. For a given regular subdivision \( T_h \) of triangles, its dual partition \( T^*_h \) is the union of the polygon. The element in the dual partition \( T^*_h \) is called control volume. The control volume is the dotted line polygon centered \( P_0 \) obtained by connecting midpoints \( M_i \)’s of edges and barycenters \( Q_j \)’s of the triangles shown in Figure 2.1. These control volumes form another partition \( T^*_h \) of \( \Omega \), which is also called dual partition.

The trial function space for velocity associated with \( T_h \) is defined as
\[
V_h = \{ v \in H^1_0(\Omega)^2 : v|_T \in P_1(T)^2, \forall T \in T_h \}.\]

The test function space \( W_h \) for velocity associated with the dual partition \( T^*_h \) is
\[
W_h = \{ w \in L^2(\Omega)^2 : w|_T \in P_0(T)^2, \forall T \in T^*_h \}.\]

Let \( Q_h \) be the finite dimensional space for pressure associated with the triangulation \( T_{2h} \)
\[
Q_h = \{ q \in L^2_0(\Omega) : q|_T \in P_0(T), \forall T \in T_{2h} \}.\]
Let $\mathcal{N}$ be a set containing all the interior nodal points associated with the partition $\mathcal{T}_h$. The operator $\gamma : H^1_0(\Omega)^2 \to W_h$ is defined by
\[
\gamma \mathbf{v}(x) = \sum_{P \in \mathcal{N}} \mathbf{v}(P) \chi_P(x), \quad \forall x \in \Omega,
\]
where $\chi_P$ is the characteristic function of the dual element associated with the node $P$.

Define
\[
(v, w)_{T_h} = \sum_{K \in \mathcal{T}_h} \int_K v \cdot w \, dx, \quad (v, w)_{\partial T_h} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} v \cdot w \, ds.
\]

We start with testing the momentum equation (1.1) by $\gamma \mathbf{v} \in W_h$,
\[
-(\nu \Delta \mathbf{u}, \gamma \mathbf{v})_{T_h} + (\mathbf{u} \cdot \nabla \mathbf{u}, \gamma \mathbf{v})_{T_h} + (\nabla p, \gamma \mathbf{v})_{T_h} = (\mathbf{f}, \gamma \mathbf{v}). \tag{2.2}
\]

Testing the continuity equation (1.2) by $q \in Q_h$ gives
\[
(\nabla \cdot \mathbf{u}, q)_{T_h} = 0. \tag{2.3}
\]

Using the integration by parts and the fact $\gamma \mathbf{v}$ is constant, the first and third terms on the left hand side of (2.2) become
\[
-(\nu \Delta \mathbf{u}, \gamma \mathbf{v})_{T_h} = -(\nu \nabla \mathbf{u} \cdot \mathbf{n}, \gamma \mathbf{v})_{\partial T_h}, \tag{2.4}
\]
\[
(\nabla p, \gamma \mathbf{v})_{T_h} = (p \mathbf{n}, \gamma \mathbf{v})_{\partial T_h}. \tag{2.5}
\]

Using the facts $\nabla \cdot \mathbf{u} = 0$ and $\gamma \mathbf{v}$ is constant and integration by parts, we have
\[
(\mathbf{u} \cdot \nabla \mathbf{w}, \gamma \mathbf{v})_{T_h} = (\mathbf{u} \cdot \mathbf{n}, \mathbf{w} \cdot \gamma \mathbf{v})_{\partial T_h}. \tag{2.6}
\]

Define
\[
\begin{align*}
a_0(\mathbf{w}, \mathbf{v}) &= -(\nu \nabla \mathbf{w} \cdot \mathbf{n}, \gamma \mathbf{v})_{\partial T_h}, \\
a_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) &= (\mathbf{u} \cdot \mathbf{n}, \mathbf{w} \cdot \gamma \mathbf{v})_{\partial T_h}, \\
c(v, q) &= (q \mathbf{n}, \gamma \mathbf{v})_{\partial T_h}, \\
b(v, q) &= (\nabla \cdot \mathbf{v}, q).
\end{align*}
\]

The finite volume scheme for the Navier-Stokes equations (1.1)–(1.3) seeks $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ such that
\[
a_0(\mathbf{u}_h, \mathbf{v}) + a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) + c(\mathbf{v}, p_h) = (\mathbf{f}, \gamma \mathbf{v}) \quad \forall \mathbf{v} \in V_h, \tag{2.6}
\]
\[
b(\mathbf{u}_h, q) = 0 \quad \forall q \in Q_h. \tag{2.7}
\]

It is clear that the solutions $(\mathbf{u}, p)$ of the Navier-Stokes equations (1.1)–(1.3) satisfy the following:
\[
a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) + c(\mathbf{v}, p) = (\mathbf{f}, \gamma \mathbf{v}) \quad \forall \mathbf{v} \in V_h, \tag{2.8}
\]
\[
b(\mathbf{u}, q) = 0 \quad \forall q \in Q_h. \tag{2.9}
\]

**Lemma 2.1.** For any $\mathbf{v}, \mathbf{w} \in H^1_0(\Omega)^2$, $q \in L^2_0(\Omega)$
\[
a_0(\mathbf{v}, \mathbf{w}) = (\nu \nabla \mathbf{v}, \nabla \mathbf{w}) + \sum_{T \in \mathcal{T}_h} (\gamma \mathbf{w} - \mathbf{w}, \nu \nabla \mathbf{v} \cdot \mathbf{n})_{\partial T}
\]
\[
+ (\nu \Delta \mathbf{v}, \mathbf{w} - \gamma \mathbf{w})_{T_h} \tag{2.10}
\]
\[
c(\mathbf{v}, q) = - (\nabla \cdot \mathbf{v}, q) + \sum_{T \in \mathcal{T}_h} ((\mathbf{v} - \gamma \mathbf{v}) \cdot \mathbf{n}, q)_{\partial T}
\]
\[
+ (\nabla q, \gamma \mathbf{v} - \mathbf{v})_{T_h}. \tag{2.11}
\]
If \( v, w \in V_h \) and \( q \in Q_h \), we have

\[
a_0(v, w) = \langle \nabla v, \nabla w \rangle, \quad c(v, q) = -(\nabla \cdot v, q) = -b(v, q).
\] (2.12)

**Proof.** The proof of (2.10) can be found in [12, 20]. (2.11) was proved in [20]. Using some calculus manipulation, (2.10) and (2.11) can also be derived directly. If \( w, v \in V_h \), i.e. \( w \) and \( v \) are linear, \( \nabla v \cdot n \) is a constant and \( \Delta v = 0 \). Also for linear function \( w \), the definition of \( \gamma \) and quadrature formula imply \( \int_K (\gamma w - w)dx = 0 \). Thus for \( w, v \in V_h \), the second and the third term in (2.10) are zero and we have \( a_0(v, w) = \langle \nabla v, \nabla w \rangle \). Similarly, we can prove \( c(v, q) = -(\nabla \cdot v, q) \) for \( v \in V_h \) and \( q \in Q_h \). \( \square \)

Let \( K \) be an element with \( e \) as an edge. It is well known that there exists a constant \( C \) such that for any function \( g \in H^2(K) \),

\[
\|g\|^2 \leq C (h_K^{-1}\|g\|_{K}^2 + h_K g_{H}^2). \tag{2.13}
\]

**Lemma 2.2.** For \( \forall v, w \in H^1(\Omega)^2 \) and \( q \in L^2(\Omega) \),

\[
|a_0(w, v)| \leq C|v|_1 \left( |w|_1 + \left( \sum_{T \in \mathcal{T}_h} h^2\|\Delta w\|_{T}^2 \right)^{\frac{1}{2}} \right), \tag{2.14}
\]

\[
|c(v, q)| \leq C|v|_1 \left( \|q\| + \left( \sum_{T \in \mathcal{T}_h} h^2\|q\|_{1,T}^2 \right)^{\frac{1}{2}} \right), \tag{2.15}
\]

**Proof.** (2.10), the Cauchy–Schwarz inequality, (2.13) and the definition of \( \gamma \) imply

\[
|a_0(w, v)| \leq \nu |w|_1 |v|_1 + \sum_{T \in \mathcal{T}_h} \nu \|\nabla v \cdot n\|_{\partial T} \|\gamma v - v\|_{\partial T} + \sum_{T \in \mathcal{T}_h} \nu \|\Delta w\|_{T} \|\gamma v - v\|_{T}
\]

\[
\leq C|v|_1 \left( |w|_1 + \left( \sum_{T \in \mathcal{T}_h} h^2\|\Delta w\|_{T}^2 \right)^{\frac{1}{2}} \right).
\]

Similarly, we can prove (2.15). \( \square \)

**Lemma 2.3.** For any \( u, v, w \in H^1(\Omega)^2 \), we have

\[
a_1(u, v, w) \leq C_i |u|_1 |v|_1 \left( |w|_1 + \left( \sum_{K \in \mathcal{T}_h} h^2|w|_{2,K}^2 \right)^{\frac{1}{2}} \right). \tag{2.16}
\]

If \( w \in V_h \), then

\[
a_1(u, v, w) \leq N_h |u|_1 |v|_1 |w|_1. \tag{2.17}
\]

**Proof.** Let \( E^*_h \) denote the union of the boundaries of the polygon \( T \) of \( T_h^* \). Define the jump of \( \gamma v \) on \( e \in E^*_h \). For the demonstration purpose, let \( e \) be the line segment \( M_2Q_1 \) in Figure 2.1. Let \( K_1 \) be quadrilateral determined by four points \( P_2, M_2, Q_2 \) and the midpoint of \( P_1 \) and \( P_2 \) and \( K_2 \) is determined by \( M_2, P_0, M_1 \) and \( Q_1 \). Define \( |\gamma v|_e = |\gamma v|_{aK_1} - |\gamma v|_{aK_2} \). Since \( u, v \in H^1(\Omega)^2 \), we have

\[
(u \cdot n, w \cdot \gamma v)_{\partial T_h^*} = \sum_{e \in E^*_h} \int_e (u \cdot n) w \cdot |\gamma v| ds.
\]
As shown in Figure 2.1, it is easy to see that

\[ [\gamma w]_e = w(P_0) - w(P_2). \]

Thus, it follows from (2.13) that

\[ \| [\gamma w]_e \|_2^2 \leq C h^2 \| \nabla w \|_e^2 \leq C (h|w|_{1,K}^2 + h^3|w|_{2,K}^2). \]

Using the Cauchy–Schwarz inequality, Corollary 5.5 in [17] and inequality above, we have

\[
\left| \sum_{e \in E_h^*} \int u \cdot n(v \cdot [\gamma w]) de \right| \leq \sum_{e \in E_h^*} \| u \|_{L^4(e)} \| v \|_{L^4(e)} \| [\gamma w] \|_{L^2(e)}
\]

\[
\leq \left( \sum_{e \in E_h^*} h_e \| u \|_{L^4(e)}^4 \right)^{\frac{1}{4}} \left( \sum_{e \in E_h^*} h_e \| v \|_{L^4(e)}^4 \right)^{\frac{1}{4}} \left( \sum_{e \in E_h^*} h_e^{-1} \| [\gamma w] \|_{L^2(e)}^2 \right)^{\frac{1}{2}}
\]

\[
\leq C |u|_1 |v|_1 \left( |w|_1 + \left( \sum_{T \in T_h} h^2 |w|_{2,T}^2 \right)^{\frac{1}{2}} \right).
\]

The inequality above and inverse inequality imply (2.17).

3. Existence and uniqueness of finite volume solutions. In this section, we will study the existence and uniqueness of the finite volume method.

3.1. Existence of finite volume solutions. In applying the Leray-Schauder fixed point theorem to the finite volume scheme (2.6) and (2.7), we introduce a discrete divergent free subspace \( D_h \) of \( V_h \) as follows:

\[ D_h = \{ v \in V_h : b(v, q) = 0, \quad \forall q \in Q_h \}. \]

Define a mesh-dependent norm

\[ \| f \|_{\star,h} = \sup_{w \in D_h} \frac{(f, v)}{|v|_1}. \quad (3.1) \]

Define \( U \) a subset of \( D_h \),

\[ U = \{ v \in D_h : |v|_1 \leq \frac{2\| f \|_{\star,h}}{\nu} \}. \]

We assume that

\[ \frac{4N_h \| f \|_{\star,h}}{\nu^2} \leq 1, \quad (3.2) \]

which implies

\[ \frac{2\| f \|_{\star,h}}{\nu} \leq \frac{\nu}{2N_h}. \quad (3.3) \]

It is easy to see that the discrete problem (2.6) and (2.7) can be reformulated as seeking \( u_h \in D_h \) satisfying

\[ a_0(u_h, v) + a_1(u_h, u_h, v) = (f, v), \quad \forall v \in D_h. \quad (3.4) \]
Let $F$ be a nonlinear map so that for each $\mathbf{w}_h \in U$, $\tilde{u}_h := F(\mathbf{w}_h)$ is given as the solution of the following linear problem:

$$
a_0(\tilde{u}_h, \mathbf{v}) + a_1(\mathbf{w}_h, \tilde{u}_h, \mathbf{v}) = (f, \mathbf{v}), \quad \forall \mathbf{v} \in D_h. \tag{3.5}
$$

Let $\mathbf{v} = \tilde{u}_h$, we have

$$
\nu |\tilde{u}_h|^2 - \mathcal{N}_h |\mathbf{w}_h|^2 |\tilde{u}_h|^2 \leq \|f \|_{*, h}|\tilde{u}_h|_1.
$$

Using (3.3), the above inequality becomes

$$
|\tilde{u}_h| \leq \frac{\|f \|_{*, h}}{\nu - \mathcal{N}_h |\mathbf{w}_h|^2} \leq \frac{2 \|f \|_{*, h}}{\nu}.
$$

Thus $F$ maps $U$ to $U$.

The map $F$ is clearly continuous and therefore is compact in the finite dimensional space $D_h$. If $\lambda > 0$ and $\mathbf{w}_h$ satisfies $\tilde{u}_h = F(\mathbf{w}_h) = \lambda \mathbf{w}_h$, then we have from (3.5) that

$$
\lambda a_0(\mathbf{w}_h, \mathbf{v}) + \lambda a_1(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}) = (f, \mathbf{v}), \quad \forall \mathbf{v} \in D_h. \tag{3.6}
$$

By choosing in (3.6) $\mathbf{v} = \mathbf{w}_h$, we come up with

$$
\lambda (a_0(\mathbf{w}_h, \mathbf{w}_h) + a_1(\mathbf{w}_h, \mathbf{w}_h, \mathbf{w}_h)) = (f, \mathbf{w}_h), \quad \forall \mathbf{v} \in D_h. \tag{3.7}
$$

Then we have from (2.12) and (2.17) that

$$
\lambda \left(\nu |\mathbf{w}_h|^2 - \mathcal{N}_h |\mathbf{w}_h|^2 \right) \leq \|f \|_{*, h} |\mathbf{w}_h|^1.
$$

It follows that

$$
\lambda \leq \frac{2 \|f \|_{*, h}}{\nu |\mathbf{w}_h|^1}.
$$

Thus, $\lambda < 1$ holds true for any $\mathbf{w}_h$ being on the boundary of the ball in $D_h$ centered at the origin with radius $\rho > \frac{2 \|f \|_{*, h}}{\nu |\mathbf{w}_h|^1}$. Consequently, the Leray-Schauder fixed point theorem implies that the nonlinear map $F$ defined by (3.5) has a fixed point $u_h$:

$$
F(u_h) = u_h
$$
in any ball centered at the origin with radius $\rho > \frac{2 \|f \|_{*, h}}{\nu}$. This fixed point $u_h$ is clearly a solution of the finite volume scheme (3.4), which in turn provides a solution of the finite volume method (2.6) and (2.7).

**Theorem 3.1.** The finite element discretization scheme (3.4) has at least one solution $u_h$ in $U$. Moreover, all the solutions of (3.4) satisfy the following estimate:

$$
|u_h| \leq \frac{2 \|f \|_{*, h}}{\nu}. \tag{3.8}
$$
3.2. A uniqueness result. The analysis here follows the idea presented in Girault and Raviart [14] on solution uniqueness for the Navier-Stokes equations. Let $\mathbf{u}_h$ and $\bar{\mathbf{u}}_h \in D_h$ be two solutions of the finite element scheme (3.4). Since both $\mathbf{u}_h$ and $\bar{\mathbf{u}}_h$ satisfy the nonlinear equation (3.4), then one has

$$a_0(\mathbf{u}_h, \mathbf{v}) + a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) = a_0(\bar{\mathbf{u}}_h, \mathbf{v}) + a_1(\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h, \mathbf{v}),$$

for all $\mathbf{v} \in D_h$. By introducing $\mathbf{e}_h = \mathbf{u}_h - \bar{\mathbf{u}}_h$, the above equation can be rewritten as

$$a_0(\mathbf{e}_h, \mathbf{v}) + a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - a_1(\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h, \mathbf{v}) = 0.$$

Observe that

$$a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - a_1(\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h, \mathbf{v}) = a_1(\mathbf{u}_h, \mathbf{e}_h, \mathbf{v}) + a_1(\mathbf{e}_h, \bar{\mathbf{u}}_h, \mathbf{v}).$$

Thus, for any $\mathbf{v} \in D_h$

$$a_0(\mathbf{e}_h, \mathbf{v}) = -a_1(\mathbf{u}_h, \mathbf{e}_h, \mathbf{v}) - a_1(\mathbf{e}_h, \bar{\mathbf{u}}_h, \mathbf{v}).$$

In particular, by letting $\mathbf{v} = \mathbf{e}_h$, we have from (2.17) and (3.8) that

$$v|\mathbf{e}_h|_H^2 \leq N_h(|\mathbf{u}_h|_1 + |\bar{\mathbf{u}}_h|^2_1) \leq \frac{4N_h\|\mathbf{f}\|_{s,h}}{\nu}|\mathbf{e}_h|^2_1.$$  

The above estimate implies obvious uniqueness under certain conditions.

**Theorem 3.2.** Assume that $\rho := \frac{4N_h\|\mathbf{f}\|_{s,h}}{\nu} < 1$ holds true and $\|\mathbf{f}\|_{s,h}$ is given by (3.1). Then the finite volume discretization scheme (3.4) has at most one solution in the discrete divergence-free subspace $D_h$.

4. Error estimates. In this section we shall establish the error estimates for the finite volume schemes (2.6)-(2.7). Our main objective is to derive an optimal-order error estimate for pressure in $L^2(\Omega)$-norm and for velocity in the discrete $H^1$-norm.

We define an operator $\Pi_1$. Since $\mathcal{T}_h$ is a fine triangulation derived from $\mathcal{T}_{2h}$, it is easy to construct an operator $\Pi_1 : H^1_0(\Omega)^2 \to V_h$ such that for $T \in \mathcal{T}_{2h}$

$$\int_e \Pi_1 \mathbf{v} \, ds = \int_e \mathbf{v} \, ds, \quad \forall \mathbf{v} \in \partial T$$

and

$$|\mathbf{v} - \Pi_1 \mathbf{v}|_{s,T} \leq C h^{1-s}|\mathbf{v}|_{s,T}, \quad \forall T \in \mathcal{T}_h, \ s = 0, 1.$$  

Using the definition of $\Pi_1$ and integration by parts, we can show that

$$b(\mathbf{v} - \Pi_1 \mathbf{v}, q) = 0, \quad \forall q \in Q_h.$$  

(4.3) and (4.2) imply the following inf-sup condition [3]:

$$\sup_{\mathbf{v} \in V_h} \frac{b(\mathbf{v}, q)}{|\mathbf{v}|_1} \geq \beta \|q\|, \quad \forall q \in Q_h.$$  

Our error analysis requires a use of the $L^2$ projection from $L^2(\Omega)$ to the finite dimensional space $Q_h$, which is denoted by $\Pi_2$. In addition, we need the following error equation: for all $\mathbf{v} \in V_h$ and $q \in Q_h$ one has

$$a_0(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) - a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) + c(\mathbf{v}, p - p_h) = 0$$

$$b(\mathbf{u} - \mathbf{u}_h, q) = 0.$$
The above error equations can be obtained from subtracting (2.6)-(2.7) from (2.8)-(2.9) and (2.12). We recall that the exact solution of (1.1)-(1.3) satisfies the following boundedness

\[ |u|_1 \leq \nu^{-1} \|f\|_1^{-1}, \]  

(4.7)

where,

\[ \|f\|_1^{-1} = \sup_{v \in [H^1_0(\Omega)]^n} \frac{\langle f, v \rangle}{|v|_1}. \]

Since \( \|f\|_1^{-1} \) and \( \|f\|_{*, h} \) are close each other when \( h \) is small, it is reasonable to assume that

\[ |u|_1 \leq 2 \|f\|_{*, h}/\nu. \]  

(4.8)

The following is an error estimate for the velocity approximation in the \( H^1 \) norm.

**Theorem 4.1.** Let \( (u, p) \) be the solution of (1.1)-(1.3) and \( (u_h, p_h) \in V_h \times Q_h \) be obtained from (2.6)-(2.7). Let

\[ \rho = \frac{4N_h \|f\|_{*, h}}{\nu^2}, \]

Assume that \( \rho < 1 \) so that the finite volume scheme (2.6)-(2.7) has a unique solution. Then, there exists a constant \( C \) independent of \( h \) such that

\[ |u - u_h|_1 \leq Ch(|u|_2 + \|p\|_1). \]  

(4.9)

**Proof.** Let

\[ \xi_h = u_h - \Pi_1 u, \quad \eta_h = p_h - \Pi_2 p \]  

(4.10)

be the error between the finite element solution \( (u_h, p_h) \) and the projection \( (\Pi_1 u, \Pi_2 p) \) of the exact solution. Denote by

\[ \xi = u - \Pi_1 u, \quad \eta = p - \Pi_2 p \]  

(4.11)

the error between the exact solution \( (u, p) \) and it projection. It follows from the error equations (4.5) and (4.6) that

\[ a_0(\xi_h, \nu) - b(\nu, \eta_h) = a_0(\xi, \nu) + c(\nu, \eta) + a_1(u, u, \nu) - a_1(u_h, u_h, \nu) \]  

(4.12)

\[ b(\xi_h, q) = b(\xi, q) = 0 \]  

(4.13)

for any \( \nu \in V_h \) and \( q \in Q_h \).

By letting \( \nu = \xi_h \) in (4.12) and \( q = \eta_h \) in (4.13), the sum of (4.12) and (4.13) gives

\[ a_0(\xi_h, \xi_h) = a_0(\xi, \xi_h) + c(\xi_h, \eta) + a_1(u, u, \xi_h) - a_1(u_h, u_h, \xi_h). \]  

(4.14)

It is easy to see that

\[ a_1(u, u, \xi_h) - a_1(u_h, u_h, \xi_h) = a_1(u, u, \xi_h) - a_1(u, u_h, \xi_h) \]

\[ + a_1(u, u_h, \xi_h) - a_1(u_h, u_h, \xi_h) \]

\[ = a_1(u, \xi) - a_1(u_h, \xi) + a_1(\xi, u_h, \xi) - a_1(\xi_h, u_h, \xi_h). \]  

(4.15)
Substituting the above equation into (4.14) yields
\[ a_0(\xi_h, \xi_h) + a_1(\mathbf{u}, \xi_h, \xi_h) + a_1(\xi_h, \mathbf{u}, \xi_h) = a_0(\xi, \xi) + c(\xi, \eta) \]
\[ + a_1(\mathbf{u}, \xi_h) + a_1(\xi, \mathbf{u}, \xi_h). \quad (4.16) \]

To estimate each term on the right-hand side of (4.16), it follows from (2.14)
\[ |a_0(\xi, \xi)| \leq C|\xi_h|_1(|\xi|_1 + h|\mathbf{u}|_2), \quad (4.17) \]
and from (2.15)
\[ |c(\xi, \eta)| \leq C|\xi_h|_1(\|\eta\| + h\|p\|_1). \quad (4.18) \]

As to the third and forth term, we have from (2.17) that
\[ |a_1(\mathbf{u}, \xi_h)| \leq C|\mathbf{u}|_1|\xi|_1|\xi_h|_1 \leq C|\xi|_1|\xi_h|_1, \quad (4.19) \]
and
\[ |a_1(\xi, \mathbf{u}, \xi_h)| \leq C|\mathbf{u}|_1|\xi|_1|\xi_h|_1 \leq C|\xi|_1|\xi_h|_1. \quad (4.20) \]

Now substituting the estimates (4.17)-(4.20) into (4.16) we obtain
\[ a_0(\xi_h, \xi_h) + a_1(\mathbf{u}, \xi_h, \xi_h) + a_1(\xi_h, \mathbf{u}, \xi_h) \leq C|\xi_h|_1(|\xi|_1 + h|\mathbf{u}|_2 + \|\eta\| + h\|p\|_1). \]

Thus, it follows from (2.12) and the above estimate that
\[ \left( \nu - \frac{4\Lambda_0^2}{\nu} \right) |\xi_h|_1^2 \leq C(|\xi|_1 + \|\eta\| + h|\mathbf{u}|_2 + h\|p\|_1)|\xi_h|_1. \quad (4.21) \]

Using the notation and the condition of Theorem 3.2, we arrive at
\[ (1 - \rho)\nu|\xi_h|_1 \leq C(|\xi|_1 + \|\eta\| + h|\mathbf{u}|_2 + h\|p\|_1). \]

Thus, the definitions of \( \Pi_1 \) and \( \Pi_2 \) imply
\[ |\xi_h|_1 \leq Ch(|\mathbf{u}|_2 + \|p\|_1). \]

The triangle inequality and the inequality above lead to the error estimates (4.9).

\[ \square \]

**Theorem 4.2.** Let \( (\mathbf{u}, p) \) be the solution of (1.1)-(1.3) and \( (\mathbf{u}_h, p_h) \in V_h \times Q_h \) be obtained from (2.6)-(2.7). Under the assumptions of Theorem 4.1, there exists a constant \( C \) independent of the mesh size \( h \) such that
\[ \|p - p_h\| \leq Ch(|\mathbf{u}|_2 + \|p\|_1). \quad (4.22) \]

**Proof.** To establish (4.22), we use the discrete inf-sup condition (4.4), the error equation (4.5) and (2.12) to obtain
\[ \|p_h - \Pi_2 p\| \leq \sup_{v \in V_h} \frac{b(v, p_h - \Pi_2 p)}{|v|_1} = \frac{-c(v, p_h - \Pi_2 p)}{|v|_1} \]
\[ = \sup_{v \in V_h} \frac{c(v, p - p_h) - c(v, p - \Pi_2 p)}{|v|_1} \]
\[ = \sup_{v \in V_h} -a_0(\mathbf{u} - \mathbf{u}_h, v) - a_1(\mathbf{u}, \mathbf{u}, v) + a_1(\mathbf{u}_h, \mathbf{u}_h, v) - c(v, p - \Pi_2 p). \]

\[ \square \]
Using (4.15), (3.8), (4.7) and (2.17), we have

\[
|a_1(u, u, v) - a_1(u_h, u_h, v)| \leq |a_1(u, u - \Pi_1 u, v) - a_1(u_h, u_h - \Pi_1 u, v) + a_1(u - \Pi_1 u, u_h, v) - a_1(u_h - \Pi_1 u, u_h, v)|
\leq C(|u - \Pi_1 u|_1 + |u_h - \Pi_1 u|_1)|v|_1.
\]

Using (2.14), (2.15), the above inequality, the definition of \(\Pi_1\) and \(\Pi_2\) and (4.9), (4.23) becomes

\[
\|p_h - \Pi_2 p\| \leq C h (\|u\|_2 + \|p\|_1).
\]

(4.22) can be derived by the triangle inequality and the equation above.

5. Numerical Examples. This section presents several numerical experiments for our finite volume methods. All the numerical examples are conducted on the Navier-Stokes equation which is defined on the unite-square domain \([0,1]^2\) with uniform triangulations. The triangulation \(T_h\) is constructed by: (1) dividing the domain into a \(n \times n\) rectangular mesh; (2) connecting the diagonal line with the negative diagonal line. Denote \(h = 1/n\) as the mesh size. Additionally, we also provide numerical results for the velocity in \(L^2\)-norm, which is expected a convergence rate of order \(O(h^2)\).

Example 1: The exact solution of \(u = (u_1, u_2)^T\) and \(p\) is assumed as

\[
\begin{align*}
  u_1(x, y) &= 10x^2(x - 1)^2y(y - 1)(2y - 1), \\
  u_2(x, y) &= -10x(x - 1)(2x - 1)y^2(y - 1)^2, \\
  p(x, y) &= 10(2x - 1)(2y - 1).
\end{align*}
\]

It can be seen that the true solution satisfies the homogeneous Dirichlet boundary condition

\[
u|_{\partial \Omega} = 0, \quad \int_\Omega p\text{d}x = 0.
\]
determine the convergence rate. The error profiles are shown in Table 5.1. For the finite volume scheme with linear polynomials, on quasi-uniform meshes of mesh size $h$, we expect a convergence of order $O(h)$ for the velocity in $H^1$ norm, and $O(h)$ for the pressure in the standard $L^2$ norm. The numerical results indicate that the convergence rate for velocity is close to 1 and that for pressure is close to 1. These are highly consistent with our theoretical conclusions.

**Example 2**: The exact solution of $u = (u_1, u_2)^T$ and $p$ is assumed as

\[
\begin{align*}
    u_1(x, y) &= -2x^2(x - 1)^2 y(y - 1)(2y - 1), \\
    u_2(x, y) &= 2y^2(y - 1)^2 x(x - 1)(2x - 1), \\
    p(x, y) &= x^2 + y^2 - 2/3.
\end{align*}
\]

It is simply to check the true solution satisfies (5.1). In this test problem, we give results for $\nu = 1$ and $\nu = 10^{-3}$. The domain is initially partitioned into triangles with $h = 1/16$. Successive uniform refinements are performed to numerically determine the convergence rate. The numerical results are shown in Table 5.2. It is observed that a convergence rate of order $O(h)$ for the velocity in $H^1$ norm, and $O(h)$ for the pressure in $L^2$ norm.

**Example 3**: Next, we set the exact solution as

\[
\begin{align*}
    u_1(x, y) &= \cos(2\pi x) \sin(2\pi y), \\
    u_2(x, y) &= -\sin(2\pi x) \cos(2\pi y), \\
    p(x, y) &= 0.
\end{align*}
\]

Since our method can easily be expended to non-homogeneous Dirichlet boundary condition, this testing problem is implemented on this case. By choosing appropriate functions $u_D$ for boundary condition, we give numerical results for $\nu = 1$ and $\nu = 10^{-3}$.

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<th>$\nu = 10^{-3}$</th>
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</table>
\[ \nu = 10^{-3} . \text{ The domain is initially partitioned into triangles with } h = 1/16. \text{ Successive uniform refinements are performed to numerically determine the convergence rate. The convergence rate of error is present in Fig 5.1. We compare the size of mesh, } h(x\text{-axis}) \text{ with error (y-axis) in the pictures. It demonstrates that the velocity in } H^1 \text{ norm is theoretically and numerically known to be of order } O(h) \text{ accuracy. Moreover, a convergence rate of the pressure approximation is } O(h), \text{ which confirms the conclusions.} \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example3.png}
\caption{Example 3 (a) the convergence rate for \( \nu = 1 \) (Left); (b) the convergence rate for \( \nu = 0.001 \) (Right).}
\end{figure}

**Example 4:** The last testing problem is tested by many researchers. We provide the results for a driven-cavity flow problem with boundary conditions:

\[
\mathbf{u}|_{\partial \Omega} = \begin{cases} 
1 & \text{if } y = 1, \\
0 & \text{else,}
\end{cases}
\]

and \( f = 0 \).

Let \( \nu = 1 \) and \( \nu = 0.05 \), Fig 5.2 shows the numerical pressure and velocity respectively. The pressure field solution (colored by velocity magnitude) has, as expected, a region of low pressure at the top left of the cavity and one of high pressure at the top right of the cavity as shown in Fig 5.2. Additionally, the velocity is present in vector field.

**REFERENCES**


Fig. 5.2. Example 4 (a) the numerical pressure and velocity for \( \nu = 1 \) (Left); (b) the numerical pressure and velocity for \( \nu = 0.05 \) (Right) on a 32\( \times \)32\( \times \)2 triangular mesh.