NEW FINITE ELEMENT METHODS IN COMPUTATIONAL FLUID DYNAMICS BY $H(\text{div})$ ELEMENTS

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Abstract. In this paper, the authors present two formulations for the Stokes problem which make use of the existing $H(\text{div})$ elements of the Raviart–Thomas type originally developed for the second-order elliptic problems. In addition, two new $H(\text{div})$ elements are constructed and analyzed particularly for the new formulations. Optimal-order error estimates are established for the corresponding finite element solutions in various Sobolev norms. The finite element solutions feature a full satisfaction of the continuity equation when existing Raviart–Thomas-type elements are employed in the numerical scheme.

Key words. finite element methods, Stokes problem

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1. Introduction. This paper is concerned with numerical solutions of incompressible fluid flow problems by finite element methods. Our objective is to introduce a finite element scheme with attention paid to the discretization of the mass continuity equation. For illustrative purposes, we show how the method works for the Stokes problem, which seeks a pair of unknown functions $(u; p)$ satisfying

\begin{align}
-\nu \Delta u + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
\n\nabla \cdot u h(x) &= 0 \quad \forall \ x \in \Omega.
\end{align}

(1.1)

(1.2)

(1.3)

where $\nu$ denotes the fluid viscosity; $\Delta$, $\nabla$, and $\nabla \cdot$ denote the Laplacian, gradient, and divergence operators, respectively; $\Omega \subset \mathbb{R}^d$ is the region occupied by the fluid; $f = f(x) \in (L^2(\Omega))^d$ is the unit external volumetric force acting on the fluid at $x \in \Omega$.

The commonly used finite element methods for the Stokes problem (1.1)–(1.3) are based on a variational equation which is obtained by testing the momentum equation (1.1) by functions in $(H^1_0(\Omega))^d$ and the continuity equation (1.2) by functions in $L^2(\Omega)$ (see section 2 for their definition). The corresponding finite element method requires a pair of finite element spaces which are conforming in $(H^1_0(\Omega))^d \times L^2(\Omega)$ and satisfy the inf-sup condition of Babuška [2] and Brezzi [3]. These constraints result in finite element approximations, denoted by $(u_h; p_h)$, which hardly satisfy the continuity equation

\begin{equation}
\nabla \cdot u_h(x) = 0 \quad \forall \ x \in \Omega.
\end{equation}

Readers are referred to [8, 19, 21] for more details regarding the approximation methods and their properties.

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The recent development in discontinuous Galerkin methods [1, 4, 5, 6, 10, 11, 13] provides new means of solving the Stokes equations numerically. However, the corresponding finite element solutions are usually totally discontinuous and fail to satisfy the continuity equation (1.4) in the classical sense [12, 22, 24, 27].

The continuity equation (1.4) requires the numerical solution $u_h$ to be a member of the Sobolev space $H(\text{div}; \Omega)$. Therefore, the discontinuous Galerkin methods [12, 22, 24, 27] appear to be noncompetitive when (1.4) needs to be satisfied. On the other hand, the $(H^1_0)^d \times L^2$ conforming finite element methods require the total continuity of $u_h$, which is too much to satisfy for (1.4). Therefore, it seems that the $H(\text{div})$ elements of Raviart–Thomas type [25, 7, 8, 17] might be good candidates for producing new numerical schemes that satisfy (1.4).

The goal of this paper is to present a method that demonstrates the use of $H(\text{div})$ elements in solving the Stokes problem. Our main contribution is on the development of a new formulation for the Stokes problem which makes use of the existing $H(\text{div})$ elements in numerical schemes. Optimal-order error estimates are derived for the resulting $H(\text{div})$ finite element approximations. In addition, two new families of $H(\text{div})$ elements are proposed and analyzed in this article.

This paper is organized as follows. In section 2, we introduce some preliminaries and notations for Sobolev spaces. A new variational formula is presented in section 3 for the Stokes problem. In section 4, we present a $H(\text{div})$ finite element method by using the variational formula developed in section 3. In section 5, we establish some optimal-order error estimates for the new finite element approximations in $H^1$ and $L^2$ norms. Finally, in section 6, we review some representatives of $H(\text{div})$ elements, followed with a detailed description of two new $H(\text{div})$ elements.

2. Preliminaries and notations. Let $D$ be any domain in $\mathbb{R}^d, d = 2, 3$. For simplicity, the method will be presented for two-dimensional problems only. An extension to higher-dimensional problems can be made formally for general polyhedral domains.

We use standard definitions for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot, \cdot)_{s, D}$, norms $\| \cdot \|_{s, D}$, and seminorms $| \cdot |_{s, D}$ for $s \geq 0$. For example, for any integer $s \geq 0$, the seminorm $| \cdot |_{s, D}$ is given by

$$|v|_{s, D} = \left( \sum_{|\alpha|=s} \int_D |\partial^\alpha v|^2 dD \right)^{\frac{1}{2}},$$

with the usual notation

$$\alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2, \quad \partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}.$$ 

The Sobolev norm $\| \cdot \|_{m, D}$ is given by

$$\|v\|_{m, D} = \left( \sum_{j=0}^{m} |v|^2_{j, D} \right)^{\frac{1}{2}}.$$

The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\| \cdot \|_D$ and $(\cdot, \cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript $D$ in the norm and inner product notation. We also use $L^2_0(\Omega)$ to denote the subspace of $L^2(\Omega)$ consisting of functions with mean value zero.
The space \( H(\text{div}; \Omega) \) is defined as the set of vector-valued functions on \( \Omega \) which, together with their divergence, are square integrable; i.e.,
\[
H(\text{div}; \Omega) = \left\{ \mathbf{v} : \mathbf{v} \in (L^2(\Omega))^2, \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}.
\]
The norm in \( H(\text{div}; \Omega) \) is defined by
\[
\|\mathbf{v}\|_{H(\text{div};\Omega)} = \left( \|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2 \right)^{1/2}.
\]

Let \( K \subset \Omega \) be a triangle or quadrilateral. For any smooth vector-valued functions \( \mathbf{w} \) and \( \mathbf{v} \), it follows from the divergence theorem that
\[
\int_K (-\Delta \mathbf{w}) \cdot \mathbf{v} dK = (\nabla \mathbf{w}, \nabla \mathbf{v})_K - \int_{\partial K} \frac{\partial w_i}{\partial x_j} \frac{\partial v_j}{\partial n} d\mathbf{s},
\]
where \( ds \) represents the boundary element, \( n_K \) is the outward normal direction on \( \partial K \), and
\[
(\nabla \mathbf{w}, \nabla \mathbf{v})_K = \sum_{i,j=1}^2 \int_K \frac{\partial w_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} dK.
\]

Let \( \tau_K \) be the tangential direction to \( \partial K \) so that \( n_K \) and \( \tau_K \) form a right-hand coordinate system. It follows from the representation
\[
\mathbf{v} = (\mathbf{v} \cdot n_K)n_K + (\mathbf{v} \cdot \tau_K)\tau_K
\]
that
\[
\frac{\partial \mathbf{w}}{\partial n_K} \cdot \mathbf{n} = \frac{\partial (\mathbf{w} \cdot n_K)}{\partial n_K}(\mathbf{v} \cdot n_K) + \frac{\partial (\mathbf{w} \cdot \tau_K)}{\partial n_K}(\mathbf{v} \cdot \tau_K).
\]

3. A variational formula. For simplicity, we let \( \nu = 1 \) for the fluid viscosity in the Stokes equation (1.1). Furthermore, we assume that \( \Omega \) is a plane polygonal domain without cracks.

Let \( T_h \) be a finite element partition of the domain \( \Omega \) with mesh size \( h \). Assume that the partition \( T_h \) is shape regular so that the routine inverse inequality in finite elements holds true (see [9]). Define the finite element spaces \( V_h \) and \( W_h \) for the velocity and pressure variables, respectively, by
\[
V_h = \{ \mathbf{v} \in H(\text{div}; \Omega) : \mathbf{v}|_K \in V_r(K) \quad \forall K \in T_h; \quad \mathbf{v} \cdot \mathbf{n}|_{\partial \Omega} = 0 \}
\]
\[
W_h = \{ q \in H^1(\Omega) : q|_K \in W_m(K) \quad \forall K \in T_h \},
\]
where \( \mathbf{n} \) is the outward normal direction on the boundary of \( \Omega \), \( V_r(K) \) is a space of vector-valued polynomials on the element \( K \) with index \( r \geq 1 \), and \( W_m(K) \) is a set of polynomials on the element \( K \) with index \( m \geq 0 \). Examples of \( V_r(K) \) and \( W_m(K) \) will be given in section 6.

To derive a weak formulation, we multiply the equation (1.1) by any \( \mathbf{v} \in V_h \) and use (2.1) to obtain
\[
\sum_{K \in T_h} \left( (\nabla \mathbf{u}, \nabla \mathbf{v})_K - \int_{\partial K} \frac{\partial \mathbf{u}}{\partial n_K} \cdot \mathbf{v} d\mathbf{s} \right) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}),
\]
where we have also used the integration by parts to deduce
\[
\int_\Omega \nabla p \cdot \mathbf{v} d\Omega = -(p, \nabla \cdot \mathbf{v}).
\]
The fact that \( \mathbf{v} \in V_h \) implies that \( \mathbf{v} \cdot \mathbf{n}_K \) is continuous across each interior boundary. Thus, it follows from (2.2) that
\[
(3.3) \quad \sum_{K \in \mathcal{T}_h} \int_{\partial_K} \frac{\partial \mathbf{u}}{\partial \mathbf{n}_K} \cdot \mathbf{v} \, ds = \sum_{K \in \mathcal{T}_h} \int_{\partial_K} \frac{\partial (\mathbf{u} \cdot \tau_K)}{\partial n_K} \mathbf{v} \cdot \tau_K \, ds.
\]
Introduce the following notation:
\[
(\nabla_h \mathbf{u}, \nabla_h \mathbf{v}) = \sum_{K \in \mathcal{T}_h} (\nabla \mathbf{u}, \nabla \mathbf{v})_K.
\]
By substituting (3.2) into (3.1) we obtain
\[
(3.4) \quad (\nabla \cdot \mathbf{u}, \mathbf{q}) = 0.
\]
We now reformulate the boundary integrals in (3.3). Let \( e \) be an interior edge shared by two elements \( K_1 \) and \( K_2 \), and let \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) be unit normal vectors on \( e \) pointing exterior to \( K_1 \) and \( K_2 \), respectively. Denote by \( \tau_1 \) and \( \tau_2 \) the two tangential directions which make the right-hand coordinate systems with \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \), respectively. We define the average \( \{ \cdot \} \) and jump \( [\cdot] \) on \( e \) for vector-valued functions \( \mathbf{w} \) as follows:
\[
\{ \mathbf{w} \} = \frac{1}{2} (\mathbf{n}_1 \cdot \nabla (\mathbf{w} \cdot \tau_1)|_{\partial K_1} + \mathbf{n}_2 \cdot \nabla (\mathbf{w} \cdot \tau_2)|_{\partial K_2}),
\]
\[
[\mathbf{w}] = \mathbf{w}|_{\partial K_1} \cdot \tau_1 + \mathbf{w}|_{\partial K_2} \cdot \tau_2.
\]
For boundary edge \( e = \partial K_1 \cap \partial \Omega \), the above two operations must be modified by
\[
\{ \mathbf{w} \} = \mathbf{n}_1 \cdot \nabla (\mathbf{w} \cdot \tau_1)|_{\partial K_1}, \quad [\mathbf{w}] = \mathbf{w}|_{\partial K_1} \cdot \tau_1.
\]
Let \( \mathcal{E}_h \) denote the union of the boundaries of all elements \( K \) in \( \mathcal{T}_h \). For sufficiently smooth \( \mathbf{u} \) (e.g., \( \mathbf{u} \in H^{2+\epsilon}(\Omega) \) for some \( \epsilon > 0 \)), it is not hard to see that
\[
\sum_{K \in \mathcal{T}_h} \int_{\partial_K} \frac{\partial (\mathbf{u} \cdot \tau_K)}{\partial n_K} \mathbf{v} \cdot \tau_K \, ds = \sum_{e \in \mathcal{E}_h} \int_{\tau} \{ \mathbf{u} \} [\mathbf{v}] \, ds.
\]
Substituting the above into (3.3) we obtain
\[
(3.5) \quad (\nabla_h \mathbf{u}, \nabla_h \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) - \sum_{e \in \mathcal{E}_h} \int_{\tau} \{ \mathbf{u} \} [\mathbf{v}] \, ds = (f, \mathbf{v}).
\]
Let \( V(h) = V_h \oplus (H^s(\Omega) \cap H^1_0(\Omega))^2 \), with \( s > \frac{3}{2} \). Denote by
\[
o_h(\mathbf{u}, \mathbf{v}) = (\nabla_h \mathbf{u}, \nabla_h \mathbf{v}) - \sum_{e \in \mathcal{E}_h} \int_{\tau} \{ \mathbf{u} \} [\mathbf{v}] \, ds
\]
and

\[ b(v, q) = (\nabla \cdot v, q) \]

two bilinear forms on \( V(h) \times V(h) \) and \( V(h) \times L_0^2(\Omega) \). With the conditions specified in this paper, it can be proved that the exact solution \((u; p)\) of the Stokes problem in 2D belongs to \( V(h) \) for some \( s > \frac{3}{2} \). Readers are referred to [20, 15, 14, 23] for details. As a result, it follows from (3.5) and (3.4) that the exact solution of the 2D Stokes problem satisfies the following variational equations:

\begin{align*}
(3.6) & \quad a_o(u, v) - b(v, p) = (f , v) \quad \forall v \in V_h, \\
(3.7) & \quad b(u, q) = 0 \quad \forall q \in W_h.
\end{align*}

However, it is not clear if the same statement can be made for the Stokes problem in three-dimensional space without assuming a smooth boundary \( \partial \Omega \) or a convex polyhedral domain \( \Omega \) [16, 18].

4. Finite element schemes. Our goal of this section is to propose two finite element schemes based on two modifications of the weak formulation (3.6)–(3.7) for the Stokes problem (1.1)–(1.3). To this end, let us introduce a symmetric bilinear form on \( V(h) \times V(h) \) as follows:

\[ a_s(w, v) = a_o(w, v) + \sum_{e \in E_h} \int_e (\alpha h_e^{-1} \| [w] \|_e - \| \varepsilon(v) \|_e [w]) \) \) \) ds,\]

where \( \alpha > 0 \) is a parameter to be determined later, and \( h_e \) is the length of the edge \( e \). For the exact solution \((u; p)\) of the Stokes problem, we clearly have

\[ a_s(u, v) = a_o(u, v) \quad \forall v \in V_h. \]

Therefore, it follows from (3.6) and (3.7) that

\begin{align*}
(4.1) & \quad a_s(u, v) - b(v, p) = (f , v) \quad \forall v \in V_h, \\
(4.2) & \quad b(u, q) = 0 \quad \forall q \in W_h.
\end{align*}

The corresponding finite element scheme for (1.1)–(1.3) seeks \((u_h; p_h)\) in \( V_h \times W_h \) such that

\begin{align*}
(4.3) & \quad a_s(u_h, v) - b(v, p_h) = (f , v) \quad \forall v \in V_h, \\
(4.4) & \quad b(u_h, q) = 0 \quad \forall q \in W_h.
\end{align*}

To investigate the properties of the above numerical scheme, we introduce two norms \( \| \cdot \|_1 \) and \( \| \cdot \| \) for the set \( V(h) \) as follows:

\begin{align*}
(4.5) & \quad \| v \|_1^2 = |v|_{1,h}^2 + \sum_{e \in E_h} h_e^{-1} \| [v] \|_e^2, \\
(4.6) & \quad \| v \|^2 = \| [v] \|_1^2 + \sum_{e \in E_h} h_e \| \varepsilon(v) \|_e^2,
\end{align*}

where \( |v|_{1,h}^2 = \sum_{K \in T_h} |v|_{1,K}^2 \) and \( \| v \|_e^2 = \int_e v \cdot v ds \).
Let $K$ be an element with $e$ as an edge. It is well known that there exists a constant $C$ such that for any function $g \in H^1(K)$
\begin{equation}
\|g\|_e^2 \leq C \left( h_K^{-1} \|g\|_K^2 + h_K \|\nabla g\|_K^2 \right).
\end{equation}
In particular, for any $v \in V_h$, we have
\begin{equation}
h_e \|\varepsilon(v)\|_e^2 \leq C \left( \|\nabla v\|_K^2 + h_K^2 \|\nabla^2 v\|_K^2 \right).
\end{equation}
The standard inverse inequality can be employed to the last term of the above inequality, yielding
\begin{equation}
h_e \|\varepsilon(v)\|_e^2 \leq C \|\nabla v\|_K^2
\end{equation}
for some constant $C$ independent of the mesh size $h$. Consequently, there is a constant $C$ such that
\begin{equation}
\|v\| \leq C_0 \|v\|_1 \quad \forall v \in V_h.
\end{equation}
The following result is concerned with the ellipticity of the bilinear form $a_s(\cdot, \cdot)$ in $V_h \times V_h$.
LEMMA 4.1. There exists a constant $\alpha_0$ independent of $h$ such that for any $v \in V_h$ we have
\begin{equation}
a_s(v, v) \geq \alpha_0 \|v\|^2,
\end{equation}
provided that $\alpha$ is sufficiently large.

Proof. It follows from the Cauchy–Schwarz inequality that there is a constant $C$ such that
\[
\left| \sum_{e \in T_h} \int_e \left( \varepsilon(w) \right) \left[ v \right] ds \right| \leq C \left( \sum_{e \in T_h} h_e \|\varepsilon(w)\|_e \right)^{\frac{1}{2}} \left( \sum_{e \in T_h} h_e^{-1} \|v\|_e \right)^{\frac{1}{2}}
\leq C_0 \|w\|_{V_h} \left( \sum_{e \in T_h} h_e^{-1} \|v\|_e \right)^{\frac{1}{2}}
\leq \frac{1}{2} \|w\|^2_{V_h} + C \sum_{e \in T_h} h_e^{-1} \|v\|^2_e,
\]
where we have used the estimate (4.9) in the second line. Using the above inequality and (4.10), we obtain
\[
a_s(v, v) = (\nabla_h v, \nabla_h v) + \alpha \sum_{e \in T_h} h_e^{-1} \int_e [v]^2 ds - 2 \sum_{e \in T_h} \int_e \varepsilon(v) \left[ v \right] ds
\geq |v|^2_{V_h} + \alpha \sum_{e \in T_h} h_e^{-1} \|v\|^2_e - \frac{1}{2} |v|^2_{V_h} - C \sum_{e \in T_h} h_e^{-1} \|v\|^2_e
= \frac{1}{2} |v|^2_{V_h} + (\alpha - C) \sum_{e \in T_h} h_e^{-1} \|v\|^2_e \geq \alpha_1 \|v\|^2_1 \geq \alpha_0 \|v\|^2,
\]
with $\alpha_1 = \min\left(\frac{1}{2}, \alpha - C\right)$ and $\alpha_0 = \alpha_1 / C_0$. For example, one may have $\alpha_0 = 1 / (2C_0)$ if the parameter $\alpha$ is chosen so that $\alpha \geq C + \frac{1}{2}$. \hfill $\square$
In the rest of the paper, we assume that the parameter $\alpha$ is chosen so that (4.11) holds true for the symmetric bilinear form $a_s(\cdot, \cdot)$. The proof of Lemma 4.1 indicates that the value of $\alpha$ depends upon the constant in the inverse inequality for finite element functions. Therefore, the value of $\alpha$ for which $a_s(\cdot, \cdot)$ is coercive is mesh-dependent. Existing results for saddle-point problems indicate that it is theoretically and computationally important to have the coercivity (4.11). Therefore, the mesh dependence of the parameter $\alpha$ makes the finite element scheme (4.3)–(4.4) conditionally interesting in practical computation.

To overcome the difficulty on the parameter selection, we introduce a second finite element scheme which is parameter-insensitive. To this end, we define a nonsymmetric bilinear form on $V(h) \times V(h)$ as follows:

$$a_{ns}(w, v) = a_o(w, v) + \sum_{e \in \mathcal{E}_h} \int_e (\alpha h_e^{-1} \|v\| + \|\varepsilon(v)\| \|w\|) \, ds.$$  

Similar to the bilinear form $a_s(\cdot, \cdot)$, for the exact solution $(u; p)$ of the Stokes problem we have

$$a_{ns}(u, v) = a_o(u, v) \quad \forall v \in V_h.$$  

Consequently, the solution of the Stokes problem satisfies the following variational equations:

(4.12) \hspace{1cm} a_{ns}(u, v) - b(v, p) = (f, v) \quad \forall v \in V_h,  

(4.13) \hspace{1cm} b(u, q) = 0 \quad \forall q \in W_h.\

Our second finite element scheme for (1.1)–(1.3) seeks $(u_h; p_h) \in V_h \times W_h$ such that

(4.14) \hspace{1cm} a_{ns}(u_h, v) - b(v, p_h) = (f, v) \quad \forall v \in V_h,  

(4.15) \hspace{1cm} b(u_h, q) = 0 \quad \forall q \in W_h.\

To see the coercivity of the bilinear form $a_{ns}(\cdot, \cdot)$, we use its definition and (4.10) to obtain

$$a_{ns}(v, v) = (\nabla_h v, \nabla_h v) + \alpha \sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e \|v\|^2 \, ds$$  

$$\geq \min(1, \alpha) \|v\|^2 \geq \min(1, \alpha) C_0^{-1} \|v\|^2,$$

where $v \in V_h$. Thus, the coercivity (4.11) holds true for the bilinear form $a_{ns}(\cdot, \cdot)$ with any value of $\alpha > 0$.

The following is a result on the boundedness of the bilinear forms $a_s(\cdot, \cdot)$ and $a_{ns}(\cdot, \cdot)$.

**Lemma 4.2.** There exists a constant $C$ independent of $h$ such that

(4.16) \hspace{1cm} |a_i(w, v)| \leq C \|w\| \|v\| \quad \forall w, v \in V(h),\

where $i = s, ns.$
where we have used the inf-sup condition for the continuous case [19, 8].

Thus, substituting (5.6) into the inequality (5.5) gives

\[ \|a_s(w, v)\| \leq C \left( \|w\|_{1,h} \|v\|_{1,h} + \left( \sum_{c \in T_h} h_c \|\varepsilon(w)\|_c^2 \right)^{1/2} \left( \sum_{c \in T_h} h_c^{-1} \|v\|_c^2 \right)^{1/2} \right. \\
+ \left. \left( \sum_{c \in T_h} h_c \|\varepsilon(v)\|_c^2 \right)^{1/2} \left( \sum_{c \in T_h} h_c^{-1} \|w\|_c^2 \right)^{1/2} \right) \]

\[ \leq C \|w\|_V \|v\|, \]

which proves the desired boundedness. \[ \square \]

5. Error estimates. The first goal of this section is to derive an optimal-order error estimate for the pressure in \( L^2(\Omega) \) and the velocity in the norm given by (4.6). The second goal is to derive an optimal-order error estimate for the velocity approximation in the \( L^2 \)-norm for the symmetric scheme (4.3)–(4.4).

Assumption 1. There exists an operator \( \Pi_h : (H_0^1(\Omega))^2 \to V_h \) such that

\[ b(v - \Pi_h v, q) = 0 \quad \forall q \in W_h. \]

In addition, the operator \( \Pi_h \) is assumed to satisfy the following:

\[ \|v - \Pi_h v\|_{s,K} \leq Ch^{l-s} |v|_{l,K} \quad \forall K \in T_h, \ s = 0, 1, \]

where the constant \( C \) depends only on the shape of \( K \) and \( 1 \leq t \leq r + 1. \)

From (5.2) and the inequality (4.7) it is not hard to see that

\[ \|v - \Pi_h v\|_1 \leq C \|v\|_1 \quad \forall v \in (H_0^1(\Omega))^2. \]

Thus, it follows from \( \|v\|_1 = |v|_1 \leq \|v\|_1 \) and the triangle inequality that

\[ \|\Pi_h v\|_1 \leq C \|v\|_1. \]

For our finite element formulations, the inf-sup condition given in Brezzi’s framework would read as follows: There exists a positive constant \( \beta \), independent of \( h \), such that

\[ \sup_{v \in V_h} \frac{b(v, q)}{\|v\|} \geq \beta \|q\| \quad \forall q \in W_h. \]

To verify (5.4), we first use the operator \( \Pi_h \) to obtain

\[ \sup_{v \in V_h} \frac{b(v, q)}{\|v\|} \geq \sup_{v \in (H_0^1(\Omega))^2} \frac{b(\Pi_h v, q)}{\|\Pi_h v\|} \leq \sup_{v \in (H_0^1(\Omega))^2} \frac{b(v, q)}{\|\Pi_h v\|}. \]

Observe that by using (5.3), and (4.10), we have for all \( v \in (H_0^1(\Omega))^2 \)

\[ \|\Pi_h v\| \leq C \|v\|_1 \leq C \|v\|_1. \]

Thus, substituting (5.6) into the inequality (5.5) gives

\[ \sup_{v \in V_h} \frac{b(v, q)}{\|v\|} \geq C^{-1} \sup_{v \in (H_0^1(\Omega))^2} \frac{b(v, q)}{\|v\|_1} \geq \beta \|q\|, \]

where we have used the inf-sup condition for the continuous case [19, 8].
5.1. Error estimates in $H^1 \times L^2$. The error analysis requires the $L^2$ projection from $L^2_0(\Omega)$ to the finite element space $W_h$, which is denoted by $Q_h$. In addition, the following error equations turn out to be useful:

\begin{align}
\alpha_s(u - u_h, v) - b(v, p - p_h) &= 0 \quad \forall v \in V_h, \\
b(u - u_h, q) &= 0 \quad \forall q \in W_h.
\end{align}

These error equations can be obtained by subtracting (4.3)–(4.4) from (4.1)–(4.2). Similar error equations hold true for the nonsymmetric scheme (4.14)–(4.15) with $\alpha_s(\cdot, \cdot)$ being replaced by $\alpha_{ns}(\cdot, \cdot)$. Now we are in a position to present an error estimate for the new finite element approximations.

**Theorem 5.1.** Let $(u; p)$ be the solution of (1.1)–(1.3) and $(u_h; p_h) \in V_h \times W_h$ be obtained from either (4.3)–(4.4) or (4.14)–(4.15). Assume that Assumption 1 holds true. Then, there exists a constant $C$ independent of $h$ such that

\begin{equation}
\|u - u_h\| + \|p - p_h\| \leq C (\|u - \Pi_h u\| + \|p - Q_h p\|).
\end{equation}

**Proof.** Let

\begin{align*}
\xi_h &= u_h - \Pi_h u, \quad \eta_h = p_h - Q_h p
\end{align*}

be the error between the finite element solution $(u_h; p_h)$ and the projection $(\Pi_h u; Q_h p)$ of the exact solution. Denote by

\begin{align*}
\xi &= u - \Pi_h u, \quad \eta = p - Q_h p
\end{align*}

the error between the exact solution $(u; p)$ and its projection. It follows from the error equations (5.7) and (5.8) that

\begin{align}
\alpha(\xi_h, v) - b(v, \eta_h) &= \alpha(\xi, v) - b(v, \eta), \\
b(\xi_h, q) &= b(\xi, q) = 0
\end{align}

for any $v \in V_h$ and $q \in W_h$. Here and in what follows of this section, $\alpha(\cdot, \cdot)$ denotes either $\alpha_s(\cdot, \cdot)$ or $\alpha_{ns}(\cdot, \cdot)$.

By letting $v = \xi_h$ in (5.10) and $q = \eta_h$ in (5.11), the sum of (5.10) and (5.11) gives

\begin{align*}
\alpha(\xi_h, \xi_h) &= \alpha(\xi, \xi_h) - b(\xi_h, \eta).
\end{align*}

Thus, it follows from the coercivity (4.11) and the boundedness (4.16) that

\begin{align*}
\alpha_0 \|\xi_h\|^2 &\leq C (\|\xi\| \|\xi_h\| + \|\eta\| \|\xi_h\|),
\end{align*}

which implies the following:

\begin{align*}
\|\xi_h\| &\leq C (\|\xi\| + \|\eta\|).
\end{align*}

The above estimate can be rewritten as

\begin{equation}
\|u_h - \Pi_h u\| \leq C (\|u - \Pi_h u\| + \|p - Q_h p\|).
\end{equation}

Now using the triangle inequality and the error estimate (5.12) we get

\begin{equation}
\|u - u_h\| \leq C (\|u - \Pi_h u\| + \|p - Q_h p\|),
\end{equation}

which completes the estimate for the velocity approximation.
It remains to estimate the pressure approximation \( p_h \). To this end, we use the discrete inf-sup condition (5.4) to obtain

\[
\| p_h - Q_h p \| \leq \frac{1}{\beta} \sup_{v \in V_h} \frac{b(v, p_h - Q_h p)}{\|v\|} = \frac{1}{\beta} \sup_{v \in V_h} \frac{b(v, p_h - p) + b(v, p - Q_h p)}{\|v\|} = \frac{1}{\beta} \sup_{v \in V_h} \frac{a(u - u_h, v) + b(v, p - Q_h p)}{\|v\|} \leq C \sup_{v \in V_h} \frac{1}{\|v\|} (\|u - u_h\| + \|p - Q_h p\|) \leq C(\|u - u_h\| + \|p - Q_h p\|),
\]

which, together with (5.13), implies that

\[
\| p_h - Q_h p \| \leq C (\|u - \Pi_h u\| + \|p - Q_h p\|).
\]

The error estimate for the pressure approximation is then completed by combining the above inequality with the standard triangle inequality.  

**5.2. An \( L^2 \)-error estimate for the velocity approximation.** Consider only the finite element approximate solutions arising from the symmetric finite element scheme. To derive an \( L^2 \)-error estimate for the velocity approximation, we seek \((w; \lambda) \in (H^1(\Omega))^2 \times L^2(\Omega)\) satisfying

\[
-\Delta w + \nabla \lambda = u - u_h \quad \text{in } \Omega,
\]

\[
\nabla \cdot w = 0 \quad \text{in } \Omega,
\]

\[
w = 0 \quad \text{on } \partial \Omega.
\]

Note that for any \((v; q) \in V(h) \times L^2(\Omega)\) the solution \((w; \lambda)\) satisfies

\[
a_s(w, v) - b(v, \lambda) = (u - u_h, v),
\]

\[
b(w, q) = 0.
\]

Assume that the Stokes problem has the \( H^2(\Omega) \times H^1(\Omega) \)-regularity property in the sense that the solution \((w; \lambda) \in (H^3(\Omega))^2 \times H^1(\Omega)\) and the following a priori estimate holds true:

\[
\|w\|_2 + \|\lambda\|_1 \leq C \|u - u_h\|.
\]

In addition, we assume that the finite element space \( V_h \) and the projection operator \( \Pi_h \) have the following property:

\[
\|w - \Pi_h w\| \leq Ch\|w\|_2.
\]

With these assumptions, it is not hard to see that there exists a constant \( C \) independent of \( h \) such that

\[
\|w - \Pi_h w\| + \|\lambda - Q_h \lambda\| \leq Ch\|u - u_h\|.
\]

It must be pointed out that the \( H^2 \times H^1 \)-regularity property assumption stated as above requires that the polygonal domain \( \Omega \) be convex. For nonconvex but smooth
domains, the regularity (5.16) can be proved to be valid. However, isoparametric elements would need to be employed in the finite element scheme in order to maintain optimal-order error estimates in either $H^1$- or $L^2$-norms.

**Theorem 5.2.** Let $(\mathbf{u}_h; p_h) \in V_h \times W_h$ and $(\mathbf{u}; p)$ be the solutions of (4.3)–(4.4) and (1.1)–(1.3), respectively. Assume that Assumption 1 and the estimate (5.17) hold true and that the Stokes problem (1.1)–(1.3) has the $H^2(\Omega) \times H^1(\Omega)$-regularity property. Then there exists a constant $C$ independent of $h$ such that

\begin{equation}
\| \mathbf{u} - \mathbf{u}_h \| \leq C h \left( \| \mathbf{u} - \Pi_h \mathbf{u} \| + \| p - Q_h p \| \right).
\end{equation}

**Proof.** By letting $\mathbf{v} = \mathbf{u} - \mathbf{u}_h$ in (5.14) we arrive at

\begin{equation}
a_s(\mathbf{u} - \mathbf{u}_h, \mathbf{w}) - b(\mathbf{u} - \mathbf{u}_h, \lambda) = \| \mathbf{u} - \mathbf{u}_h \|^2.
\end{equation}

Notice that

\begin{equation}
b(\mathbf{u} - \mathbf{u}_h, \lambda) = b(\mathbf{u} - \mathbf{u}_h, \lambda - Q_h \lambda)
\end{equation}

and

\begin{equation}
a_s(\mathbf{u} - \mathbf{u}_h, \mathbf{w}) = a_s(\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \Pi_h \mathbf{w}) + a_s(\mathbf{u} - \mathbf{u}_h, \Pi_h \mathbf{w})
\end{equation}

\begin{equation}
= a_s(\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \Pi_h \mathbf{w}) + b(\Pi_h \mathbf{w}, p - p_h)
\end{equation}

\begin{equation}
= a_s(\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \Pi_h \mathbf{w}) + b(\Pi_h \mathbf{w} - \mathbf{w}, p - p_h).
\end{equation}

Substituting (5.21) and (5.22) into (5.20) we obtain

\begin{equation}
\| \mathbf{u} - \mathbf{u}_h \|^2 = a_s(\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \Pi_h \mathbf{w}) + b(\Pi_h \mathbf{w} - \mathbf{w}, p - p_h) - b(\mathbf{u} - \mathbf{u}_h, \lambda - Q_h \lambda).
\end{equation}

Thus,

\begin{equation}
\| \mathbf{u} - \mathbf{u}_h \|^2 \leq C \left( \| \mathbf{u} - \mathbf{u}_h \| + \| p - p_h \| \right) \left( \| \mathbf{w} - \Pi_h \mathbf{w} \| + \| \lambda - Q_h \lambda \| \right).
\end{equation}

Substituting (5.18) into the above estimate we obtain

\begin{equation}
\| \mathbf{u} - \mathbf{u}_h \|^2 \leq C h \left( \| \mathbf{u} - \mathbf{u}_h \| + \| p - p_h \| \right) \| \mathbf{u} - \mathbf{u}_h \|,
\end{equation}

which implies that

\begin{equation}
\| \mathbf{u} - \mathbf{u}_h \| \leq C h \left( \| \mathbf{u} - \mathbf{u}_h \| + \| p - p_h \| \right).
\end{equation}

The above inequality and the error estimate (5.9) imply

\begin{equation}
\| \mathbf{u} - \mathbf{u}_h \| \leq C h \left( \| \mathbf{u} - \Pi_h \mathbf{u} \| + \| p - Q_h p \| \right).
\end{equation}

This completes the proof of the theorem. \qed

**6. Examples of $H(\text{div})$ elements.** Let us recall that the error estimates established in section 5 are based on the following three properties:

B1. $V_h \subset H(\text{div}; \Omega),$

B2. Assumption 1 and the estimate (5.17) as described in section 5, and

B3. the $H^2 \times H^1$-regularity property assumption for the Stokes problem.
The last property (B3) is required only for the $L^2$-error estimate for the velocity approximation. This means that any finite element pair $V_h \times W_h$ satisfying properties B1–B2 is applicable for the formulations presented earlier in this manuscript.

Denote by $P_h(K)$ the space of polynomials of degree $\leq k$ and

$$
P_{h,k_1,k_2}(K) = \left\{ p(x_1, x_2) : p(x_1, x_2) = \sum_{0 \leq i \leq k_1, 0 \leq j \leq k_2} a_{ij} x_1^i x_2^j \right\}.
$$

$P_{h,k_1,k_2,k_3}(K)$ is defined similarly in three-dimensional spaces. Define $Q_h(K)$ as follows:

$$
Q_h(K) = \begin{cases} 
P_{h,k}(K) & \text{for } d = 2, \\
P_{h,k,k}(K) & \text{for } d = 3. 
\end{cases}
$$

Observe that the finite element pair $V_h \times W_h$ is constructed from local elements $V_r(K)$ and $W_m(K)$ as described in section 3. Therefore, it suffices to specify the local pair $V_r(K) \times W_m(K)$ for each example to be presented.

6.1. Existing elements. All of the existing $H(\text{div})$ elements designed for the second-order elliptic problems (e.g., see [25, 8, 7, 17, 19]) satisfy properties B1–B2, except the estimate (5.17) for the lowest-order Raviart–Thomas element on triangles and quadrilaterals. Therefore, there are plenty of finite element spaces applicable to the new formulation of the Stokes problem. For illustrative purposes, we mention three examples. Readers are referred to the book by Brezzi and Fortin [8] for more examples of the $H(\text{div})$ element.

6.1.1. Raviart–Thomas elements on triangles or tetrahedra: $RT_k(K)$. Let $k \geq 1$ be any integer. For any triangular or tetrahedral element $K$, the local element $V_r(K) \times W_m(K)$ is defined by

$$
V_h(K) = (P_h(K))^d \oplus \mathbf{x}P_h(K), \quad W_h(K) = P_h(K),
$$

where $d = 2$ if $K$ is a triangle and $d = 3$ if $K$ is a tetrahedron. The projection operator $\Pi_h$ satisfying all of the required properties is given locally on each element $K$. For example, the restriction of $\Pi_h$ on the element $K$, denoted by $\Pi_K$, is defined as follows:

$$
\int_{\partial K} (v - \Pi_K v) \cdot n q ds = 0 \quad \forall q \in P_h(\partial K),
$$

$$
\int_K (v - \Pi_K v) \cdot q dK = 0 \quad \forall q \in (P_{h-1}(K))^d, k \geq 1.
$$

6.1.2. BDM elements on triangles or tetrahedra: $BDM_k(K)$ [8]. Let $k \geq 1$ be any integer. For any triangular or tetrahedral element $K$, the local element $V_r(K) \times W_m(K)$ is defined by setting $r = m + 1 = k$ and

$$
V_h(K) = (P_h(K))^d, \quad W_h(K) = P_{k-1}(K).
$$

On a triangular element $K$, the local projection operator $\Pi_K : (H^1(K))^2 \to V_h(K)$ is defined by

$$
\int_{\partial K} (v - \Pi_K v) \cdot n q ds = 0 \quad \forall q \in P_h(\partial K),
$$

$$
\int_K (v - \Pi_K v) \cdot \nabla q dK = 0 \quad \forall q \in P_{k-1}(K),
$$

$$
\int_K (v - \Pi_K v) \cdot \text{curl}(b_K q) dK = 0 \quad \forall q \in P_{k-2}(K), k \geq 2.
$$
where $b_K$ is the bubble function defined on $K$. On a tetrahedral element $K$, the corresponding local projection $\Pi_K$ is given by

$$
\int_{\partial K} (\mathbf{v} - \Pi_K \mathbf{v}) \cdot n q ds = 0 \quad \forall q \in P_k(\partial K),
$$

$$
\int_K (\mathbf{v} - \Pi_K \mathbf{v}) \cdot \nabla q dK = 0 \quad \forall q \in P_{k-1}(K),
$$

$$
\int_K (\mathbf{v} - \Pi_K \mathbf{v}) \cdot q dK = 0 \quad \forall q \in \Phi_k(K),
$$

where

$$
\Phi(K) = \{ \phi \in (P_k(K))^3 : \nabla \cdot \phi = 0, \ \phi \cdot n = 0 \text{ on } \partial K \}.
$$

### 6.1.3. BDM elements on quadrilaterals: $BDM_{kr}(K)$

It is sufficient to describe the element on the unit square. Let $k \geq 1$ be any integer. The local element $V_r(K) \times W_m(K)$ is defined by

$$
V_r(K) = (P_k(K))^2 \oplus \text{curl}(x_1^{k+1}, x_2) \oplus \text{curl}(x_1 x_2^{k+1}),
$$

$$
W_{k-1}(K) = P_{k-1}(K).
$$

On the unit square element $K$, the local projection operator $\Pi_K : (H^1(K))^2 \to V_k(K)$ is defined by

$$
\int_{\partial K} (\mathbf{v} - \Pi_K \mathbf{v}) \cdot n q ds = 0 \quad \forall q \in P_k(\partial K),
$$

$$
\int_K (\mathbf{v} - \Pi_K \mathbf{v}) \cdot \mathbf{w} dK = 0 \quad \forall \mathbf{w} \in (P_{k-2}(K))^2, k \geq 2.
$$

### 6.1.4. Error estimates for the existing elements

Recall that the velocity $V_h$ and the pressure space $W_h$ are defined, respectively, by

$$
V_h = \{ \mathbf{v} \in H(\text{div}; \Omega) : \mathbf{v}|_K \in V_r(K) \quad \forall K \in T_h ; \mathbf{v} \cdot n|_{\partial \Omega} = 0 \}
$$

and

$$
W_h = \{ q \in L^2_0(\Omega) : q|_K \in W_m(K) \quad \forall K \in T_h \}.
$$

For the existing $H(\text{div})$ elements listed above, we have $V_r(K) = RT_k(K), BDM_k(K)$, or $BDM_{kr}(K)$ and $W_m(K) = P_k(K), P_{k-1}(K)$, or $P_{k-1}(K)$, respectively. The projection operator $\Pi_h$ is given by

$$
(\Pi_h \mathbf{v})|_K = \Pi_K (\mathbf{v}|_K).
$$

The definition of $\Pi_h$ implies that

$$
b(\mathbf{v} - \Pi_h \mathbf{v}, q) = 0 \quad \forall q \in W_h.
$$

Furthermore, it has been proved in [8] that (5.2) and (5.17) hold true for $\Pi_h$ defined in (6.3). Therefore, properties B1–B2 are well justified.

Let $Q_h$ be the $L^2$ projection from $L^2_0(\Omega)$ to $W_h$. It is not hard to see that $W_h$ has the following local approximation properties: For $BDM_k(K)$ and $BDM_{kr}(K)$

$$
|p - Q_h p|_{s,K} \leq Ch^{k-s} |p|_{k,K} \quad \forall K \in T_h, \ s = 0, 1.
$$
and for $RT_k(K)$

\[(6.6) \quad |p - Q_h p|_{s,K} \leq Ch^{k+1-s} |p|_{k+1,K} \quad \forall K \in T_h, \ s = 0, 1.\]

The constant $C$ in (6.5)–(6.6) depends only on $k$ and the shape of $K$.

The following result follows from (5.17), (6.5)–(6.6), and Theorems 5.1 and 5.2.

**Proposition 6.1.** Let $(u; p)$ be the solution of (1.1)–(1.3) and $(u_h; p_h) \in V_h \times W_h$ be obtained from either (4.3)–(4.4) or (4.14)–(4.15). Assume that $(u; p) \in (H^{k+1}(\Omega))^2 \times H^1(\Omega)$ for some $1 \leq t \leq k$. Then there exists a constant $C$ independent of $h$ such that for $BDM_h(K)$, $BDM_{[k]}(K)$, and $RT_k(K)$

\[(6.7) \quad \|u - u_h\| + \|p - p_h\| \leq C h^t (\|u\|_{t+1} + \|p\|_{t}).\]

Furthermore, if the $H^2 \times H^1$-regularity property holds true for the Stokes problem, then there is a constant $C$ such that the finite element approximation $(u_h; p_h)$ from the symmetric formulation has the following error estimate:

\[\|u - u_h\| \leq C h^{t+1} (\|u\|_{t+1} + \|p\|_{t}).\]

We comment that the above error estimates hold true for all of the $H(\text{div})$ elements listed in [8].

**6.2. New elements.** Stability and accuracy are two main factors in the construction of new finite elements. For the variational schemes presented in this paper, the stability part is realized by a combination of the inf-sup condition and the coercivity for the corresponding bilinear forms. The accuracy part is characterized by Assumption 1 and a balanced pressure space.

The symmetric formulation has the following error estimate:

\[\|u - u_h\| \leq C h^{t+1} (\|u\|_{t+1} + \|p\|_{t}).\]

Furthermore, if the $H^2 \times H^1$-regularity property holds true for the Stokes problem, then there is a constant $C$ such that the finite element approximation $(u_h; p_h)$ from the symmetric formulation has the following error estimate:

\[\|u - u_h\| \leq C h^{t+1} (\|u\|_{t+1} + \|p\|_{t}).\]

We comment that the above error estimates hold true for all of the $H(\text{div})$ elements listed in [8].

**6.2.1. A new element on rectangles: $NE1_k(K)$:** We illustrate the construction of the new $NE1_k(K)$ element on the unit square $K = [0, 1] \times [0, 1]$. Let $k \geq 1$ be any integer. We define local elements $V_r(K) \times W_m(K)$ by

\[V_r(K) = (Q_k(K))^2, \quad W_{m-1}(K) = Q_{k-1}(K).\]

For the first component of $v = (v_1, v_2)$, we define an operator $\Pi_{K,1} : H^1(K) \to Q_k(K)$ as follows:

\[(6.8) \quad \int_e (v_1 - \Pi_{K,1} v_1) \phi ds = 0 \quad \forall \phi \in P_k(e), \ e = \text{west, east},\]

\[(6.9) \quad \int_K (v_1 - \Pi_{K,1} v_1) \psi dK = 0 \quad \forall \psi \in P_{k-2,K}(K),\]
where $e = \text{west}$ means that $e$ is the west edge (i.e., $e = \{(0, x_2) : x_2 \in [0, 1]\}$) of the unit square; the east edge is defined accordingly.

The system (6.8) involves exactly $2(k + 1)$ linear equations and (6.9) involves $(k - 1)(k + 1)$ linear equations. The total number of equations is given by

$$2(k + 1) + (k - 1)(k + 1) = (k + 1)^2,$$

which is the same as the total number of degrees of freedom for a polynomial in $Q_k(K)$. The following proposition shows that the linear systems (6.8) and (6.9) uniquely determine the projection $\Pi_{K,1}v_1$.

**Proposition 6.2.** Let $v \in Q_k(K)$ be such that

$$\int_E v\phi ds = 0 \quad \forall \phi \in P_k(e), e = \text{west, east}, \tag{6.10}$$

$$\int_K v\psi dK = 0 \quad \forall \psi \in P_{k-2,k}(K). \tag{6.11}$$

Then we must have $v \equiv 0$.

**Proof.** The condition (6.10) implies that $v = 0$ at the east and west edges of the unit square $K$. Thus, there is a polynomial $g = g(x_1, x_2) \in P_{k-2,k}(K)$ such that $v = x_1(1 - x_1)g$. Substitute $v = x_1(1 - x_1)$ into (6.11), and then let $\psi = g$. It follows that $g \equiv 0$. This shows that $v \equiv 0$.

The projection of the second component of $v$, denoted by $\Pi_{K,2}v_2$, can be defined in a similar fashion. The local projection operator is then given by

$$\Pi_K v = (\Pi_{K,1}v_1, \Pi_{K,2}v_2).$$

It is not hard to show that such a defined projection satisfies all of the conditions required in the previous sections. As a result, the element $NE1_k(K)$ can be used to approximate the Stokes problem.

**6.2.2. A new element on cubes: $NE2_k(K)$**. Again, we shall describe details only on the unit cube $K = [0, 1]^3$. Let $k \geq 1$ be an integer. A straightforward extension of the $NE1_k(K)$ to three-dimensional space is given by

$$V_k(K) = (Q_k(K))^3, \quad W_{k-1}(K) = Q_{k-1}(K).$$

Our goal here is to show that the above extension actually works. To this end, it suffices to construct a projection operator $\Pi_K$ which satisfies the required properties.

Let $v = (v_1, v_2, v_3) \in (H^1(\Omega))^3$ be a vector-valued function. For each component $v_i$, we define its projection to $Q_k(K)$ as follows:

$$\int_{e_i} (v_i - \Pi_{K,i}v_i)\phi ds = 0 \quad \forall \phi \in Q_k(e_i), \tag{6.12}$$

$$\int_K (v_i - \Pi_{K,i}v_i)\psi dK = 0 \quad \forall \psi \in P_{k_1,k_2,k_3}(K), \tag{6.13}$$

where $e_i = \{(x_1, x_2, x_3) : x_j \in [0, 1], j \neq i; x_i = 0 \text{ or } 1\}$ are the two faces of the cube $K$ which are orthogonal to the $x_i$-axis, and $k_i = k - 2$, $k_j = k$ for $j \neq i$.

There are $2(k + 1)^2$ linear equations from the condition (6.12) and $(k - 1)(k + 1)^2$ linear equations from the condition (6.13). The total number of linear equations is then given by

$$2(k + 1)^2 + (k - 1)(k + 1)^2 = (k + 1)^3,$$
which is the same as the total number of degrees of freedom for a polynomial in the space \( V_k(K) \).

A similar argument as in the previous subsection for \( NE1_k(K) \) can be applied to show that the projection \( \Pi_{K,i}v_i \) is uniquely determined by (6.12) and (6.13). Furthermore, the local projection given by

\[
\Pi_K v = (\Pi_{K,1}v_1, \Pi_{K,2}v_2, \Pi_{K,3}v_3)
\]

can be verified to satisfy all of the properties required by the convergence theory developed in previous sections for the new finite element methods.

6.2.3. Another new element on rectangles: \( NE3_k(K) \). Again for simplicity, we shall describe the new element on the unit square \( K = [0,1]^2 \). This element will be a simplified version of \( NE1_k(K) \) but with the same order of accuracy.

Let \( k \geq 1 \) be any integer, and define

\[
V_k(K) = (P_k(K) \oplus \{x_1x_2^k\}) \times (P_k(K) \oplus \{x_2x_1^k\}),
\]

\[
W_{k-1}(K) = P_{k-1}(K).
\]

For the first component of \( v = (v_1,v_2) \), we define its projection \( \Pi_{K,1}v_1 \in P_k(K) \oplus \{x_1x_2^k\} \) by using the following equations:

\[
\int_e (v_1 - \Pi_{K,1}v_1) \phi ds = 0 \quad \forall \phi \in P_k(e), \ e = \text{west, east},
\]

\[
\int_K (v_1 - \Pi_{K,1}v_1) \psi dK = 0 \quad \forall \psi \in P_{k-2}(K).
\]

There are \( 2(k+1) \) equations from the condition (6.14) and \( \frac{1}{2}(k-1)k \) equations from the condition (6.15). The total number of linear equations is given by

\[
2(k+1) + \frac{1}{2}(k-1)k = \frac{1}{2}(k+1)(k+2) + 1,
\]

which is the same as the total number of degrees of freedom for functions in the space \( P_k(K) \oplus \{x_1x_2^k\} \). Using the same technique as in the analysis for \( NE1_k(K) \), it can be proved that \( \Pi_{K,1}v_1 \) is uniquely determined by (6.14) and (6.15). The projection of the second component of \( v \) can be determined in a similar way. The resulting local projection \( \Pi_K v = (\Pi_{K,1}v_1, \Pi_{K,2}v_2) \) satisfies all of the properties required in the convergence theory.

6.2.4. Error estimates for the new elements. First, we define the velocity space \( V_h \) by

\[
V_h = \{ v \in H(\text{div}; \Omega) : v|_K \in V_r(K), \ \forall K \in \mathcal{T}_h; v \cdot n|_{\partial \Omega} = 0 \}
\]

and the pressure space \( W_h \) by

\[
W_h = \{ q \in L_0^2(\Omega) : q|_K \in W_m(K), \ \forall K \in \mathcal{T}_h \},
\]

where \( V_r(K) = NE1_k(K), NE2_k(K), \) or \( NE3_k(K) \) and \( W_m(K) = Q_{k-1}(K), Q_{k-1}(K), \) or \( P_{k-1}(K), \) respectively. For any \( v \in (H_0^1(\Omega))^d, \) with \( d = 2, 3 \), define \( \Pi_h v \in V_h \) by

\[
(\Pi_h v)|_K = \Pi_K v \quad \forall K \in \mathcal{T}_h,
\]
where \( \Pi_K \) is the corresponding local projection operator on each element. From the construction of \( \Pi_K \), it is easy to see that it is indeed true that \( \Pi_h \mathbf{v} \in V_h \), and, moreover, one has

\[
(6.19) \quad b(\mathbf{v} - \Pi_h \mathbf{v}, q) = 0 \quad \forall q \in W_h
\]

and that properties B1–B2 are satisfied for the three new elements \( NE_{1k}(K) \), \( NE_{2k}(K) \), and \( NE_{3k}(K) \). Similar to Proposition 6.1, we have the following convergence estimates.

**Proposition 6.3.** Let \( (\mathbf{u}, p) \) be the solution of (1.1)–(1.3) and \( (\mathbf{u}_h, p_h) \in V_h \times W_h \) be obtained from either (4.3)–(4.4) or (4.14)–(4.15) by using the new elements described in this subsection. Assume that \( (\mathbf{u}, p) \in (H^{t+1}(\Omega))^d \times H^t(\Omega) \) for some \( \frac{1}{2} < t \leq k \). Then there exists a constant \( C \) independent of \( h \) such that

\[
\|\mathbf{u} - \mathbf{u}_h\| + \|p - p_h\| \leq Ch^t(\|\mathbf{u}\|_{t+1} + \|p\|_t),
\]

and for the symmetric formulation we also have

\[
\|\mathbf{u} - \mathbf{u}_h\| \leq Ch^{t+1}(\|\mathbf{u}\|_{t+1} + \|p\|_t),
\]

provided that the \( H^2 \times H^1 \)-regularity property holds true for the Stokes problem.

We point out that, unlike the existing \( H(\text{div}) \) elements, the new \( H(\text{div}) \) elements described in this section do not yield numerical velocities that satisfy the continuity equation (1.4) in the classical sense. However, the numerical approximations arising from the new elements indeed conserve mass locally on each element.

**REFERENCES**


