New class of finite element methods: weak Galerkin methods

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Consider second order elliptic problem:

\[-\nabla \cdot a \nabla u = f, \quad \text{in } \Omega \quad (1)\]
\[u = 0, \quad \text{on } \partial \Omega. \quad (2)\]

Testing (1) by \( v \in H^1_0(\Omega) \) gives

\[-\int_{\Omega} \nabla \cdot a \nabla uvdx = \int_{\Omega} a \nabla u \cdot \nabla vdx - \int_{\partial \Omega} a \nabla u \cdot n vds = \int_{\Omega} fvdx. \]

\[(a \nabla u, \nabla v) = (f, v),\]

where \((f, g) = \int_{\Omega} fgdx\)
Continuous Finite element methods

Weak form: find \( u \in H^1_0(\Omega) \) such that
\[
(a \nabla u, \nabla v) = (f, v), \quad \forall v \in H^1_0(\Omega).
\]

Given \( \mathcal{T}_h \), let \( V_h \subset H^1_0(\Omega) \) be a finite element space.
\[
V_h = \{ v \in H^1_0(\Omega); \ v|_T \in P_k(T), \ T \in \mathcal{T}_h \}.
\]

Continuous Finite element method: find \( u_h \in V_h \) such that
\[
(a \nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v \in V_h,
\]
Find \( u_h \in V_h \) such that

\[
(a \nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in V_h.
\]

Let \( V_h = \text{Span}\{\phi_1, \cdots, \phi_n\} \) and \( u_h = \sum_{j=1}^{n} c_j \phi_j \), then

\[
\sum_{j=1}^{n} (a \nabla \phi_j, \nabla \phi_i) c_j = (f, \phi_i), \quad i = 1, \cdots, n.
\]

Simple formulations and many fewer unknowns.
Modern techniques in scientific computing

- $hp$ adaptive technique.
- Hybrid mesh.

The continuous finite element method is not compatible to these techniques.
Difficult to construct continuous elements

- $C^0$ element: high order element.
- $C^1$ element: Argyris element, polynomial with degree 5.

Solution? Discontinuous elements
Discontinuous Galerkin finite element methods

Interior penalty finite element method

\[ \sum_{T} (a \nabla u_h, \nabla v_h)_{\mathcal{T}} - \sum_{e} ((a \{ \nabla u_h \}, [v_h])_e - \sigma (a \{ \nabla v_h \}, [u_h])_e) \]

\[ + \alpha \sum_{e} h^{-1}([u_h], [v_h])_e = (f, v_h). \]

Weakly over penalized interior penalty finite element method

\[ \sum_{T} (a \nabla u_h, \nabla v_h)_{\mathcal{T}} + \alpha \sum_{e} h^{-3} (\Pi_0 [u_h], \Pi_0 [v_h])_e = (f, v_h). \]
LDG methods

Find $q_h \in V_h$, $u_h \in W_h$ such that

$$(a^{-1}q_h, v) + (\nabla \cdot v, u_h)_{T_h} - \langle \hat{u}_h, v \cdot n \rangle_{\partial T_h} = 0, \quad \forall v \in V_h$$

$$(q_h, \nabla w)_{T_h} - \langle \hat{q}_h \cdot n, w \rangle_{\partial T_h} = (f, w), \quad \forall w \in W_h,$$

where

$$\hat{u}_h = \{u_h\} - \beta \cdot [u_h],$$

$$\hat{q}_h = \{q_h\} + \beta [q_h] - \alpha [u_h].$$

Mixed Hybrid method and the HDG method

**Mixed hybrid finite element method:** find \( q_h \in V_h, \ u_h \in W_h \) and \( \hat{u}_h \in M_h \) such that

\[
(a^{-1}q_h, v) - (\nabla \cdot v, u_h)_{T_h} + \langle \hat{u}_h, v \cdot n \rangle_{\partial T_h} = 0, \quad \forall v \in V_h
\]

\[
(\nabla \cdot q_h, w)_{T_h} = (f, w), \quad \forall w \in W_h,
\]

\[
\langle \mu, q_h \cdot n \rangle_{\partial T_h} = 0, \quad \forall \mu \in M_h.
\]

**HDG method:** find \( q_h \in V_h, \ u_h \in W_h \) and \( \hat{u}_h \in M_h \) such that

\[
(a^{-1}q_h, v) - (\nabla \cdot v, u_h)_{T_h} + \langle \hat{u}_h, v \cdot n \rangle_{\partial T_h} = 0, \quad \forall v \in V_h
\]

\[
(\nabla \cdot q_h, w)_{T_h} + \tau \langle u_h - \hat{u}_h, w \rangle_{\partial T_h} = (f, w), \quad \forall w \in W_h
\]

\[
\langle \mu, q_h \cdot n \rangle_{\partial T_h} + \tau \langle u_h - \hat{u}_h, \mu \rangle_{\partial T_h} = 0, \quad \forall \mu \in M_h.
\]


What is the right finite element formulation when discontinuous approximation functions are used

Starting point of the finite element methods: weak form of the PDE:

\[(a \nabla u, \nabla v) = (f, v).\]

The right finite element formulations should be similar to the corresponding weak forms of the PDEs.

For discontinuous approximation function \(v\), \(\nabla v\) is not well defined.

A natural choice of finite element formulation for discontinuous elements should have the form

\[(a \nabla_w u_h, \nabla_w v_h) + s(u_h, v_h) = (f, v_h),\]

where \(s(u_h, v_h)\) is a stabilizer without tuning parameters.
Weak Galerkin finite element methods

• Define weak function $v = \{v_0, v_b\}$ such that

$$v = \begin{cases} v_0, & \text{in } T^0 \\ v_b, & \text{on } \partial T \end{cases}$$

Define weak Galerkin finite element space

$$V_h = \{v = \{v_0, v_b\} : v_0|_T \in P_j(T^0), v_b \in P_\ell(e), e \subset \partial T, v_b = 0 \text{ on } \partial \Omega \}.$$

• Define a discrete weak gradient $\nabla_w v \in [P_r(T)]^d$ for $v \in V_h$ on each element $T$:

$$(\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + \langle v_b, q \cdot n \rangle_{\partial T}, \quad \forall q \in [P_r(T)]^d.$$
Weak Galerkin finite element methods

Find \( u_h \in V_h \subset L^2(\Omega) \) such that for any \( v_h \in V_h \)

\[
(a \nabla_w u_h, \nabla_w v_h) + \sum_T h_T^{-1} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T} = (f, v_h).
\]

**Theorem.** Let \( u_h \) be the solution of the WG method associated with local spaces \((P_k(T), P_k(e), [P_{k-1}(T)]^d)\),

\[
h \| Q_h u - u_h \| + \| Q_h u - u_h \| \leq C h^{k+1} \| u \|_{k+1}.
\]
$v = \{v_0, v_b\} \in P_j(T) \times P_\ell(e)$ and $\nabla_w v \in [P_r(T)]^d$.

Examples:

• Choose local functions spaces $(P_j(T), P_\ell(e), [P_r(T)]^d)$. Let $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^d)$, the WG method:

$$ (a \nabla_w u_h, \nabla_w v) = (f, v_0). $$

• Choose $v_b$ to be fixed: $v = \{v_0, v_b\} = \{v, \{v\}\}$.

$$ (\nabla_w v, q)_T = - (v, \nabla \cdot q)_T + \langle \{v\}, q \cdot n \rangle_{\partial T}. $$
Modified weak Galerkin method: Find $u_h \in V_h$ such that

$$(a\nabla_w u_h, \nabla_w v) + \sum_e h_e^{-1}\langle[u_h], [v]\rangle_e = (f, v), \quad \forall v \in V_h.$$ 


$$\sum_T (a\nabla u_h, \nabla v)_T - \sum_e ((a\{\nabla u_h\}, [v])_e - \sigma(a\{\nabla v\}, [u_h])_e)$$

$$+ \alpha \sum_e h^{-1}([u_h], [v])_e = (f, v).$$
Simplifying existing methods

The elliptic equation in mixed form:

\[
(a^{-1}q, v) - (\nabla u, v) = 0 \\
(\nabla \cdot q, w) = (f, w).
\]

LDG method:

\[
(a^{-1}q_h, v) + (\nabla \cdot v, u_h)_{\mathcal{T}_h} - \langle \hat{u}_h, v \cdot n \rangle_{\partial \mathcal{T}_h} = 0, \\
(q_h, \nabla w)_{\mathcal{T}_h} - \langle \hat{q}_h \cdot n, w \rangle_{\partial \mathcal{T}_h} = (f, w), \\
\hat{u}_h = \{u_h\} - \beta [u_h], \\
\hat{q}_h = \{q_h\} + \beta [q_h] - \alpha [u_h].
\]

LDG method in term of weak derivatives with $\beta = 0$:

\[
(a^{-1}q_h, v) - (\nabla w u_h, v)_{\mathcal{T}_h} = 0, \\
-(\nabla w \cdot q_h, w)_{\mathcal{T}_h} - \alpha \langle [u_h], w n \rangle_{\partial \mathcal{T}_h} = (f, w).
\]
HDG method:

\[(a^{-1}q_h, v) - (\nabla \cdot v, u_h)_T + \langle \hat{u}_h, v \cdot n \rangle_{\partial T} = \langle g, v \cdot n \rangle_{\partial \Omega}, \quad \forall v \in V_h\]

\[-(q_h, \nabla w)_T + \langle \hat{q}_h \cdot n, w \rangle_{\partial T} = (f, w), \quad \forall w \in W_h,\]

\[\langle \mu, \hat{q}_h \cdot n \rangle_{\partial T} = 0, \quad \forall \mu \in M_h,\]

\[\hat{q}_h = q_h + \tau(u_h - \hat{u}_h)n.\]

HDG method in term of weak derivative:

\[(a^{-1}q_h, v) - (\nabla_w u_h, v)_T = 0,\]

\[-(q_h, \nabla_w w)_T - \tau\langle u_h - \hat{u}_h, w - \hat{w} \rangle_{\partial T} = (f, w).\]

HDG method and weak Galerkin method are equivalent when \(a\) is piecewise constant. They are not equivalent in general.
Schur Complement of the WG formulation

The WG method: find \( u_h = \{u_0, u_b\} \in V_h \) such that

\[
a(u_h, v) = (f, v_0), \quad \forall \ v = \{v_0, v_b\} \in V_h
\]

For \( u_h = \{u_0, u_b\} \), solve for \( u_0 \) in term of \( u_b \) on \( T \)

\[
a(u_h, v) = (f, v_0)_T, \quad \forall v = \{v_0, 0\} \in V_h,
\]

Denote \( u_0 = D(u_b, f) \). Find \( u_b \) satisfies

\[
a(\{D(u_b, f), u_b\}, v) = 0, \quad \forall v = \{0, v_b\} \in V_h.
\]

The system above: symmetric, positive definite, fewer unknowns.
Weak form of the Stokes equations: find 
\((u, p) \in [H^1_0(\Omega)]^d \times L^2_0(\Omega)\) that for all 
\((v, q) \in [H^1_0(\Omega)]^d \times L^2_0(\Omega)\)

\[
\begin{align*}
(\nabla u, \nabla v) - (\nabla \cdot v, p) &= (f, v) \\
(\nabla \cdot u, q) &= 0.
\end{align*}
\]

Weak Galerkin method: find \((u_h, p_h) \in V_h \times W_h\) such that for all 
\((v, q) \in V_h \times W_h\)

\[
\begin{align*}
(\nabla_w u_h, \nabla_w v) + s(u_h, v) - (\nabla_w \cdot v, p_h) &= (f, v) \\
(\nabla_w \cdot u_h, q) &= 0.
\end{align*}
\]
The weak form of the Stokes equations: seeking \( u \in H^2_0(\Omega) \) satisfying
\[
(\Delta u, \Delta v) = (f, v), \quad \forall v \in H^2_0(\Omega),
\]
Weak Galerkin finite element method: seeking \( u_h \in V_h \) satisfying
\[
(\Delta_w u_h, \Delta_w v) + s(u_h, v) = (f, v), \quad \forall v \in V_h.
\]
Summary

- The weak Galerkin finite element methods represent advanced methodology for handling discontinuous approximation functions.
- The weak Galerkin methodology provide a general framework for deriving new methods and simplifying the existing methods.
- Simple formulations imply easy analysis and easy applications.